# Analyzing Randomly Placed Multiple Antennas for MIMO Wireless Communication

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***Abstract*—We present an analytical approach for determining the signal-to-noise-ratio (SNR) of $m$ multiple antennas in the line-of-sight case. The antennas are placed at random positions within a disc of given diameter $d$. We characterize the angular signal strength with three sectors: the main beam, the side beams and an area of white Gaussian noise. The SNR and the sector angles depend on $d$, $m$, and the wavelength $\lambda$.**

**It turns out that for randomized antenna positions the analysis can be reduced to the analysis of a random geometric walk in two dimensions. The angle of the main beam is approximately $\lambda/d$ with a SNR proportional to $\sqrt{m}$. For the side beams the SNR is proportional to sinc$(2\alpha d/\lambda)$ where $\alpha$ denotes the angle deviating from the target. The range of the side beams is limited to an approximate angle of $\lambda/d\sqrt{m}$. Beyond this angle we observe average white Gaussian noise.**

***Index Terms*—MIMO, 2D random walk, signal-to-noise ratio, channel capacity**

## I. Introduction

Among the many merits of multiple-input multiple-output (MIMO) technology for wireless networks there are higher data rates, larger communication range, and reduced energy consumption. These are achieved by the joint use of antennas for senders and receivers. The senders adapt at the antennas the signal carrier amplitudes and phase shifts resulting in a spatially depending attenuation of the signal. The receivers can produce the symmetric behavior by demodulation. This can be exploited to increase the signal strength for a desired sender-receiver connection while reducing the interfering noise to other network nodes.

Signal processing in such systems with its practical usage for angular perception of sensor arrays for sonar and wireless networks have been studied over decades [1]. One field of interest is the antenna placement [2], [3], [4], [5] for beamforming or for an appropriate selection of an antennas subset for the best beamforming [6], [7], [8], [9], [10], [11], [12], [13], [14].

Little is known about how the antenna positions at the node influence the quality of a MIMO transmission. It is known that the textbook example [15] of a linear equidistant antenna array can be analyzed, but performs badly when the communication partner is located perpendicular to the linear array. Then the sending or receiving beam has a wide angle. In practice this is only a minor problem since the antenna geometry is neither used to predict nor solely determines the channel matrix $\mathbf{H}$[1]. MIMO benefits from multipath propagation and for the MIMO transmission the channel matrix is measured. Little research has been made where the relation between the channel matrix $\mathbf{H}$ and the antenna geometry has been considered. For example, Foschini et al. assume in [16] the transmitted signal components to be idealized statistically independent gaussians caused by a large number of scatterers to deduce a channel capacity linear to the number of multiple antennas. This obfuscates the geometry of the original multiple antennas.

In this paper we consider line-of-sight propagation and discuss the geometric properties of beamforming by reference to the geometry of multiple antennas. We choose a random uniform placement of antennas in a disc of diameter $d$ to overcome the shortcomings of the linear equidistant placement. This matches a practical scenario where antennas are non uniformly attached to a device or even are flexible installed. Furthermore, we do not consider the channel matrix $\mathbf{H}$ but directly compute the signal strengths in a given direction. This way, we derive bounds which generally describe the signal beam angle with respect to the antenna geometry parameter $d$, the wavelength $\lambda$, the number of sender/receiver antennas $m_s$, resp. $m_r$, and the distance between sender and receiver. We classify the angles into three classes, the main beam which is useful for transmission or reception, the side beams which may cause interferences with other nodes, and the random noise, which adds only little noise to the system, see Figure 1.

The capacity $C$ of a communication channel between a sender-receiver pair is determined by both the SNR at the receiver and the properties of the channel respectively channel matrix $\mathbf{H}$, e.g. $C = \log_2(1 + \mathrm{SNR} \cdot \mathbf{hh}^*)\,\mathrm{Bit/s/Hz}$ [17] for a SIMO system with single input and multiple output with channel vector $\mathbf{h}$ for multiple receiver antennas and its conjugate transpose $\mathbf{h}^*$. With our model, the noise of the SNR relation influenced by parallel transmissions can be described more precisely. The beamforming gain specially dominates the capacity improvement of multiple antennas when the angular spread between communication partners is too small and the channel matrix $\mathbf{H}$ has a low rank [18].

## II. Related Work

Krim and Viberg summarize in [1] the development in signal processing and consider uniform linear and circular

[1]Each entry $h_{ij}$ in the channel matrix $\mathbf{H}$ denotes the channel gain between the $i$-th transmit antenna and $j$-th receive antenna.

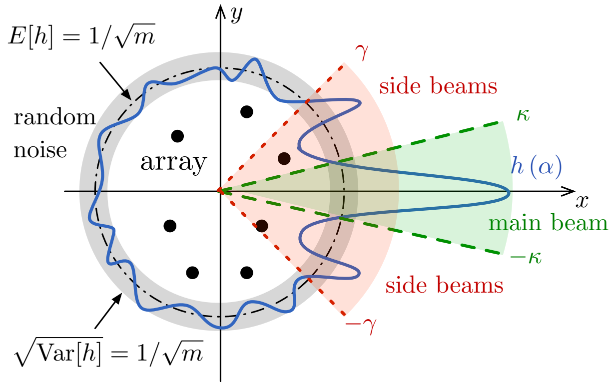



Fig. 1. Example for seven randomly placed antennas (black dots) in a disc (white disc in the center) showing the signal strength $h(\alpha)$ around the disc (bold blue line) depending on the angle. Sending direction is $\alpha = 0$ (along the $x$-axis) with the main beam in angle range $[-\kappa, \kappa]$, side beams within angle range $[-\gamma, \gamma]$, and average white Gaussian noise in other directions.

array geometries. Indeed they focus on sensor arrays but the geometric derivations can be applied to antenna arrays in the same way. Tse and Viswanath describe in [15] multiple antenna arrays limited to uniform linear arrays and derive the channel capacity under the the Rayleigh fading model.

In [19] Pollock et al. extend the MIMO channel capacity calculation with parameters of the antenna positioning and the angular spread. In theory the capacity grows linearly with the number of antennas for Rayleigh fading channels. But they show for realistic scenarios that the capacity is significant lower because of insufficient antenna spacing and angular spread.

In [20] Foschini and Driesen investigate the impact of the array geometry of multiple antennas on the channel matrix. They consider line-of-sight propagation for linear arrays, arrays with antennas spread along an arc and uniform circular arrays. Additionally, they analyze linear arrays located on a street with two reflecting building walls alongside the street. They also compute the capacity for Ricean channels.

The authors of [5] categorize placing the transmit and receive antennas of a MIMO system for highest capacity as fundamental design issue. With Particle Swarm Optimization, they search for the antenna placement with highest channel capacity which they derive from the properties of the channel matrix $\mathbf{H}$. They include scattering objects as isotropic radiators in their model which are uniformly distributed. Olgun et al. present numerical results for the placement of antennas in two-dimensional plane and three-dimensional space where each antenna can be placed at $100^2$ respectively $100^3$ discrete points. They compare the capacity of arrays created with Particle Swarm Optimization (PSO) with the capacity of uniform linear arrays (ULA) and uniform circular arrays (UCA). For up to six antennas per array, PSO outperforms UCA and UCA outperforms ULA. For more than six antennas per array, PSO barley improves the capacity level of UCA and the authors conclude that uniform circular arrays are a good design option.

## III. Physical Model

First, we introduce mathematical notations used in this paper. We use the Euler's constant $e$, the constant $\pi$, the imaginary number $j = \sqrt{-1}$, and the speed of light $c \approx 3 \cdot 10^8$ m/s. Bold written variables with small letters e.g. $\mathbf{u}$ define two-dimensional vectors representing the $(x, y)$-coordinates in the plane. Respectively, bold written capital letters are matrices, e.g. channel matrix $\mathbf{H}$. The Euclidean distance between two points $\mathbf{u}$ and $\mathbf{v}$ in the plane is stated as $\|\mathbf{u}, \mathbf{v}\| = \sqrt{(\mathbf{u}_x - \mathbf{v}_x)^2 + (\mathbf{u}_y - \mathbf{v}_y)^2}$. $\mathrm{P}[X = x]$ denotes the probability that a random variable $X$ has the value $x$, $\mathrm{E}[X]$ is the expectation of $X$, $\mathrm{Var}[X]$ is the variance of $X$, and $\sqrt{\mathrm{Var}[X]}$ is the standard deviation of $X$ respectively. We use the Landau notation $\mathcal{O}(\cdot)$ and $\Omega(\cdot)$ to describe asymptotic behavior.

The following model is presented in [15] and adapted to our needs. The physical input-output-model of a MISO channel (Multiple Input Single Output) with $m$ transmit antennas and a single receiver antenna with output signal $y$ is

$$y = \sum_{i=1}^{m} h_i \cdot x_i + w$$

where $x_i$ is the input signal of the $i$-th transmit antenna. We assume that all transmit antennas emit the same input signal $x = x_i$ with the same amplitude but different phase shifts (to produce beamforming). The parameter $w$ defines white noise which is Gaussian distributed $w \sim \mathcal{N}(0, \sigma^2)$ with variance $\sigma^2$. The baseband channel gain $h_i$ for the $i$-th transmit antenna is

$$h_i = \frac{a_i}{m} \cdot e^{-\frac{j2\pi}{\lambda} \cdot \|\mathbf{u}[i], \mathbf{v}\|}.$$

The signal is attenuated with factor $a_i$ during the transmission. This includes a radiation pattern of the antenna in space where the signal fades with factor $\|\mathbf{u}[i], \mathbf{v}\|^{-1}$ for distance $\|\mathbf{u}[i], \mathbf{v}\|$ between the transmit antenna located at $\mathbf{u}[i]$ and target position $\mathbf{v}$. Respectively, the power per area fades with $\|\mathbf{u}[i], \mathbf{v}\|^{-2}$ in the far field, a.k.a. path loss[2]. We norm $h_i$ with $1/m$ where $m$ is the number of transmit antennas. The signal is demodulated on a carrier frequency $f$ and is propagated in space with speed of light $c \approx 3 \cdot 10^8$ m/s resulting in a wavelength $\lambda = c/f$. Common carrier frequencies of the 802.11 IEEE standards for wireless communication are around 5 Ghz and 2.4 GHz respectively carrier wavelengths 6 cm and 12.5 cm. Due to different path lengths $\|\mathbf{u}[i], \mathbf{v}\|$ between transmit antenna $\mathbf{u}[i]$ and a target position $\mathbf{v}$, the signals of the different transmit antennas $\mathbf{u}[i]$ are time-displaced at $\mathbf{v}$. We only consider the

[2]In practice, the path loss exponent depends highly on the environment e.g. obstacles absorbing the energy. We also neglect the path loss induced by the antenna geometry.

phase shift for the carrier wavelength $\lambda$ expressed by a complex value $e^{j\kappa}$ with phase angle $\kappa$. The phase angle $\kappa$ effects a modulo computation of the time for propagation $\|\mathbf{u}[i], \mathbf{v}\| / c$ between a transmit antenna $\mathbf{u}[i]$ and position $\mathbf{v}$ modulo the period $1/f = \lambda/c$. The superpositioned time-displaced signals of the antennas $\mathbf{u}[i]$ produce a spatial attenuation of the signal. This effect is called *beamforming* where the signal is strong in certain spatial beams and attenuated otherwise.

## IV. Random Antenna Placement

Let $u$ denote a network node with $m$ multiple antennas and let $\mathbf{u}\,[i]$ denote the $(x, y)$-position in the plane of the $i$-th antenna with $i \in [1, m]$. The antennas are placed independently and uniformly at random on a disc in the plane with diameter $d$ and $\mathbf{u}$ denotes the centroid of the disc. Thus, the maximum distance between two antennas placed at positions $\mathbf{u}\,[i]$ and $\mathbf{u}\,[\ell]$ is

$$d \quad \geq \quad \max_{i,\ell \in [1,m]} \|\mathbf{u}\,[i]\,, \mathbf{u}\,[\ell]\|\,.$$

We require that diameter $d$ is at least $\Omega\,(\lambda \cdot \sqrt{m})$ for carrier wavelength $\lambda$ and the number of antennas $m$.

We assume that two communication nodes $u$ and $v$ are in far distance compared to $d$, i.e. $\|\mathbf{u}, \mathbf{v}\| \gg d$. So, we estimate the received signal strengths from all sender antennas $\mathbf{u}\,[i]$ ($i \in [1, m]$) at position $\mathbf{v}$ as

$$y \quad = \quad \frac{a}{m} \sum_{i=1}^{m} e^{-\frac{j2\pi}{\lambda} \cdot \|\mathbf{u}[i], \mathbf{v}\|} x_i + w\ . \tag{1}$$

For this input-output model, we optimize the signal strength of a given sending antenna array in the next Section IV-A, e.g. Eq. 1 is resolved to $y = a \cdot x + w$ for the target position. Then, we identify the angular ranges of the main and side beams where such a strong signal can be received (Section IV-B). In the remaining angle range, we estimate the attenuated signal by average white Gaussian noise (Section IV-C).

The superpositioned output signal strength of the multiple antennas of Equation 1 is calculated from the addition of complex values which can be represented as two-dimensional vectors. Assuming unit power at all antennas all vectors have unit length and only the angles differ caused by the phase shift. Figure 2 shows an example for the different cases with maximum signal strength in the main beam where the signals of all antennas arrive with the same phase angle. In the side beams, which are spatially close to the main beam (compare Figure 1), the phase angles are again highly correlated. Otherwise for a radiation angle differing more from the target direction ($\alpha \geq \gamma$, see Sec. IV-C) we observe random phase angles resulting into a strong attenuated signal which we denote as average white Gaussian noise (dark grey).

### A. Set up beamforming for arbitrary placed antennas

To achieve maximum signal strength of multiple input antennas at a given target $y = a \cdot x + w$, one can adjust the phases of the multiple antennas in such a way that they are highly correlated at the target. For that, the signal is delayed at

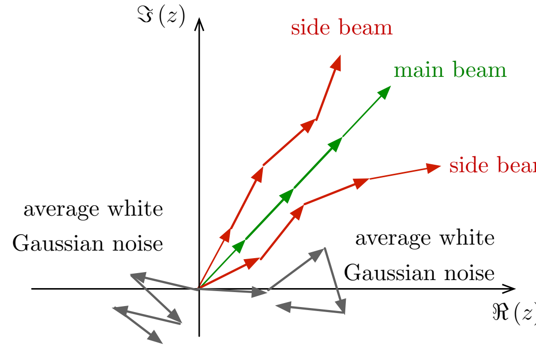



Fig. 2. Example for calculating the signal strength $h\,(\alpha)$ with $z = \left(\frac{a}{m} \cdot \sum_i e^{j\beta_i}\right)$ for different angles $\alpha$. The signal of all antennas has the same signal amplitude $\frac{a}{m}$ (same complex vector length) but different phase angles $\beta_i$.

the input antennas in such a way that the delay time plus the transmission time from each antenna to the target is the same. For an explanation consider the antenna array in Figure 3 with two antennas positioned at $\mathbf{u}[1]$ and $\mathbf{u}[2]$ where $\mathbf{u}[1]$ has a longer path to the target in direction $\phi$ than $\mathbf{u}[2]$.

Reflections can be seen as additional signal sources but with an attenuated and time shifted signal. The running time from an antenna via a reflecting obstacle to the target and the line-of-sight time from the antenna to the target depend on each other. Thus, we cannot adjust both signals in any way that both arrive at the same time at the target. A reflected signal always arrives delayed at the target in comparison to the line-of-sight signal and produces additional noise. We will only consider the line-of-sight case in the latter.

Now we will define how to set up the phase shift on an antenna array with arbitrary antenna positions to gain maximum signal strength in a certain target direction. This theoretic approach neglects reflection and similar effects, which is the reason why in existing MIMO systems the measured channel matrix is the target for optimization. Here, we consider the simplified scenario of line-of-sight communication in the plane.

We assume the node-to-target distance to be much larger than the maximum distance of each nodes' antennas. So, the antennas' rays reach the target nearly as parallel lines [3]. Assume a virtual antenna in the centroid of the antenna array $\mathbf{u}$ with phase $0$. The target direction of this array is $\phi$. The signal $x$ is shifted in time at all antennas such that it reaches the target in direction $\phi$ at the same time. The time shift of each antenna $\mathbf{u}[i]$ can be derived from a geometrical argument, i.e. the difference of distances between $\mathbf{u}$ and $\mathbf{u}[i]$ and the target divided by the speed of light. For distant targets this time shift can be approximated by a function depending only on the sender antenna positions and the sending angle $\phi$. In Figure 3, this time shift is shown for the positions $\mathbf{u}[1]$ and

[3]When we apply both assumptions of far distant communication nodes and no angular spread between multiple antennas to a MIMO channel, the channel matrix $\mathbf{H}$ has rank 1. Hence, beamforming, respectively the SNR, dominates then the channel capacity.

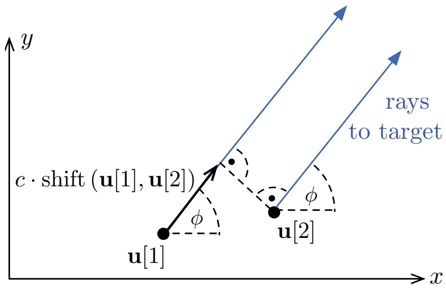



Fig. 3. Phase shift between reference antenna $u_1$ and antenna $u_2$, target direction $\phi$, spatial shift $c \cdot \mathrm{shift}\left(\mathbf{u}_1, \mathbf{u}_2, \phi\right)$ between both antennas towards target ($c$ is the speed of light).

$\mathbf{u}[2]$ with label $c \cdot \mathrm{shift}\left(\mathbf{u}[1], \mathbf{u}[2]\right)$.

Let vector $\mathbf{v}$ with an arbitrary non-zero length and angle $\phi$ describe a ray towards the target with

$$\mathbf{v} \quad := \quad |\mathbf{v}| \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}.$$

We can use the scalar projection of $(\mathbf{u}_i - \mathbf{u})$ and $\mathbf{v}$ and angle $\phi$ to compute the spatial shift

$$c \cdot \mathrm{shift}\left(\mathbf{u}, \mathbf{u}[i], \phi\right) \quad = \quad \left(\mathbf{u}[i] - \mathbf{u}\right) \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

where $c$ denotes the speed of light. The time shift or delay is then

$$\begin{aligned} &\mathrm{shift}\left(\mathbf{u}, \mathbf{u}[i], \phi\right) \\ &\quad = \frac{1}{c}\left(\mathbf{u}_x - \mathbf{u}[i]_x\right)\cos\phi + \frac{1}{c}\left(\mathbf{u}_y - \mathbf{u}[i]_y\right)\sin\phi. \end{aligned} \tag{2}$$

W.l.o.g. the centroid is at $(0, 0)$. So, the phase shift of the signal for antenna $\mathbf{u}[i]$ for a communication partner in direction $\phi$ is then

$$\mathrm{shift}\left(\mathbf{u}[i], \phi\right) = \frac{1}{c}\mathbf{u}[i]_x \cos\phi + \frac{1}{c}\mathbf{u}[i]_y \sin\phi. \tag{3}$$

We assume that we send the same signal long enough and therefore we can represent time shifts as phase shifts. Producing beamforming with these phase shifts will give the signal strength $y = a \cdot x + w$ at the communication partner.

### B. *Characterization of the main beam and side beams*

If the phases of signal $x$ at multiple antennas are optimized for a transmission in one particular direction, the phase angles around that sending angle are still correlated and not random and the superpositioned output signal strength fades from the maximum signal strength $y = a \cdot x + w$ with increasing angle difference. We claim that the signal strength is proportional to the maximum signal strength in an angle range $\alpha \in [-\kappa, \kappa]$ (see Figure 1).

**Theorem 1** *The angle range of the main beam tends to* $\Theta(\lambda/d)$ *when* $d/\lambda$ *grows to infinity.*

To analyze the signal strength in the angle range $\alpha \in [-\kappa, \kappa]$ around the target direction $\phi$, we shift in time the input signal $x$ on the multiple antennas with the delay time of Equation 3 for angle $\phi$ to optimize the signal strength towards angle $\phi$. Furthermore, we insert an additional delay for angle $(\phi + \alpha)$ into the equation to calculate the signal strength in direction $(\phi + \alpha)$ which we want to analyze. The signal strength is then

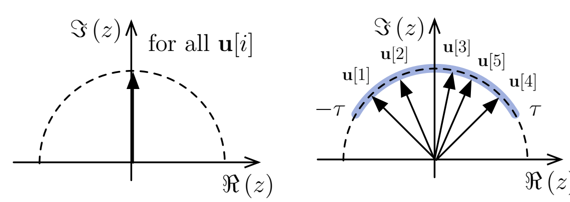
$$h\left(\phi, \alpha\right) \quad = \quad \frac{a}{m} \cdot \sum_{i=1}^{m} e^{-j2\pi f \mathrm{shift}(\mathbf{u}[i], \phi)} \cdot e^{j2\pi f \mathrm{shift}(\mathbf{u}[i], \phi+\alpha)}$$

where the additive Gaussian noise $w$ is only omitted in the equation for a simpler presentation. Notice that if the receiver is in the target direction $\phi$ with $\alpha = 0$, both delays in the previous equation eliminate each other and we get the maximum signal strength. W.l.o.g. we set the target direction $\phi = 0$ resulting into

$$h\left(\phi = 0, \alpha\right) \quad = \quad \frac{a}{m} \cdot \sum_{i=1}^{m} e^{-j\frac{2\pi}{\lambda}\left(u[i]_x \cdot (\cos\alpha - 1) + u[i]_y \sin\alpha\right)} \tag{4}$$

$$\approx \quad \frac{a}{m} \cdot \sum_{i=1}^{m} e^{-j\frac{2\pi}{\lambda}u[i]_y \cdot \alpha}. \tag{5}$$

The last approximation in Equation 5 uses $\sin\alpha = \alpha$, $\cos\alpha = 1$ for small $\alpha$. For $\alpha = 0$ all antennas have the same phase angle 0 and varying $\alpha$ rotates the phase angles with different speed depending on the vertical position $\mathbf{u}[i]$ of the $i$-th antenna. At direction $\alpha$ the phase angle is then limited




(a) $\alpha = 0$ (b) $\alpha > 0$

Fig. 4. Illustration how complex vectors change their phase angle when varying angle of radiation $\alpha$ with (a) $\alpha = 0$ and all phase angles equal 0 and (b) $\alpha > 0$ and phase angles in the range $[-\tau, \tau]$.

by range $[-\tau, \tau]$ with

$$\tau \quad := \quad \frac{2\pi}{\lambda} \cdot d \cdot \alpha.$$

Based on our assumption of randomly distributed antennas we further assume equally distributed phase angles in range $[-\tau, \tau]$. Thus, we can estimate the sum by an integral over the range $[-d,d]$ resulting into

$$\begin{aligned}
& h_2(\phi=0,\alpha) \\
\approx\ & \frac{a\cdot m}{m}\cdot\frac{\int_{-d}^{d} e^{-j\frac{2\pi\alpha}{\lambda}y}\,\mathrm{d}y}{2\tau} \\
=\ & -\frac{a\cdot m}{m}\cdot\left[\frac{e^{-j\frac{2\pi\alpha}{\lambda}y}}{j\frac{2\pi\alpha}{\lambda}\cdot 2d}\right]_{y=-d}^{d} \\
=\ & -a\cdot\left[\frac{\cos\left(-\frac{2\pi\alpha}{\lambda}y\right)+j\sin\left(-\frac{2\pi\alpha}{\lambda}y\right)}{j\frac{2\pi\alpha}{\lambda}\cdot 2d}\right]_{y=-d}^{d} \\
=\ & a\cdot\frac{\sin\left(\frac{2\pi\alpha}{\lambda}\cdot d\right)}{\frac{2\pi\alpha}{\lambda}\cdot d} \\
=\ & a\cdot\mathrm{sinc}\left(\frac{2d}{\lambda}\cdot\alpha\right).
\end{aligned}$$

According to the sinc$(\cdot)$-function, the main beam is bounded by angle region

$$\alpha\in[-\kappa,\kappa] \text{ with } \kappa=\frac{\lambda}{2d}.$$

For instance, when the antennas are placed on a disc with radius $2\lambda$ and a typical wavelength $\lambda=12.5$ cm the main beam has the range $[-\kappa',\kappa']$ with angle $\kappa'\approx 7$ degree.

Besides, the main beam at $\alpha=0$ there are recurrent side beams at the maxima of the sinc$(\cdot)$-function. The signal gain of these side beams is limited according to the sinc$(\cdot)$-function by $\lambda/(2\pi\cdot d\cdot\alpha)$. In the next section we will show that the side beams are within angle range $[-\gamma,\gamma]$ with $\gamma\approx\frac{\lambda}{d}\sqrt{m}$.

### C. Average white Gaussian noise produced by multiple antennas

Now we will analyze the random noise of a sender with an angle outside of the side beams. Recall that for $\mathbf{u}[i]$ chosen uniformly from a disc of diameter $d$

$$h(0,\alpha) = \frac{a}{m}\cdot\sum_{i=1}^{m} e^{-j\frac{2\pi}{\lambda}(u[i]_x\cdot(\cos\alpha-1)+u[i]_y\sin\alpha)}. \quad (6)$$

Let $\beta_i = \frac{2\pi}{\lambda}\left(u[i]_x\cdot(\cos\alpha-1)+u[i]_y\sin\alpha\right)$ denote the random variable of the phase angle. Figure 5 shows the distribution of this random variable for $\lambda=2\pi$. Note that for growing $d/\lambda$ the range of the random variable increases linearly. Let $[-\ell,\ell]$ denote the range of $\beta_i$. The maximum value for $\ell$ is $\frac{4\pi}{\lambda}d$, for small $\alpha$. We can approximate $\ell$ by $\ell\approx\frac{2\pi}{\lambda}\alpha d$, since $\sin\alpha\approx\alpha$ and $\cos\alpha\approx 1$ for small $\alpha$.

We approximate this random variable by a uniform distribution over $[-\ell,\ell]$. The following Lemma shows that for $\ell\geq 2\pi\sqrt{m}$ the absolute value of the random variable $H$ has an expectation of at most $\frac{a}{\sqrt{m}}$ and a standard deviation of $2\frac{a}{\sqrt{m}}$. For uniform $\beta_i$ the signal strength can be reduced to the length of a 2-dimensional geometric random walk with unit steps in the plane. Here, the diameter $d$ of the disk for antenna placement has to be in the order of $\Omega(\lambda\cdot\sqrt{m})$ that the phase angles $\beta_i$ are uniformly distributed.

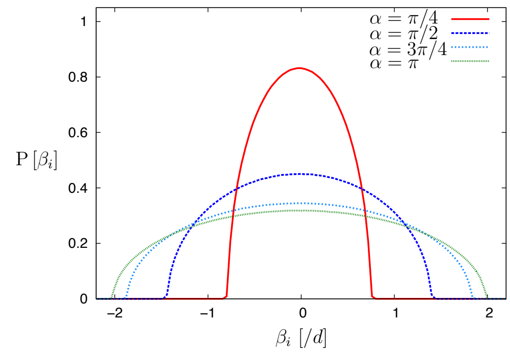



Fig. 5. Probability distribution of phase angle $\beta_i$ in $\sum_i e^{-j\beta_i}$ of Equation 4 where for $\lambda=2\pi$ for illustration purposes only.

**Lemma 1** *Let $H_{m,\ell} := \sum_{i=1}^{m} e^{j\beta_i}$ for uniformly distributed $\beta_i$ from $[-\ell,\ell]$.*

1) *$E\left[|H_{m,\ell}|^2\right]=m$ for $\ell\neq 0$ which are a multiples of $2\pi$*
2) *$E\left[|H_{m,\ell}|\right]=\mathcal{O}(\sqrt{m})$ for all $\ell\geq 2\pi\sqrt{m}$*
3) *$E\left[|H_{m,\ell}|^2\right]=\mathcal{O}(m)$ for all $\ell$*

*Proof:* If $\ell>0$ is a multiple of $2\pi$ all angles are uniformly distributed. A two-dimensional geometric walk starts at $s_0=(0,0)$ and continues for $m$ steps at points $s_1,\ldots,s_m$ where each step $s_{i+1}-s_i$ has unit length and an independently randomly chosen direction. Such geometric walks have been studied for a long time, see [21] and the following theorem is well-known and its proof can be found in standard textbooks.

Each vector $s_i$ with unit length 1 and direction $\beta_i$ can be represented as complex value $s_i = e^{j\beta_i} = j\sin\beta_i+\cos\beta_i$ where $j=\sqrt{-1}$. The distance between start and end point of the random walk is then the vector length of the sum of all vectors.

$$h = \left|\sum_{j=1}^{n} e^{j\beta_i}\right|$$

Let $\bar{h}$ denote the complex conjugate of $h$.

$$\begin{aligned}
|h|^2 &= h\cdot\bar{h} \\
&= \underbrace{\sum_{i=1}^{m} e^{j\beta_i}}_{h}\cdot\underbrace{\sum_{k=1}^{m} e^{-j\beta_k}}_{\bar{h}} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m} e^{j(\beta_i-\beta_k)} \\
&= m+\sum_{i=1}^{m}\sum_{\substack{k=1,\\ i\neq k}}^{m} e^{j(\beta_i-\beta_k)}
\end{aligned}$$

For each index tuple $(i,k)$ with $i\neq k$ there exists a symmetric $(k,i)$ with the negated imaginary value.

$$\forall i\neq k: \quad \Im\left(e^{j(\beta_i-\beta_k)}\right)+\Im\left(e^{j(\beta_k-\beta_i)}\right)=0$$

So, we get only a sum of real numbers.

$$\sum_{i=1}^{n}\sum_{\substack{k=1,\\ i\neq k}}^{n} e^{j(\beta_i-\beta_k)} = \sum_{i=1}^{n}\sum_{\substack{k=1,\\ i\neq k}}^{n} \cos(\beta_i-\beta_k)$$

We have assumed that angles $\beta_i \in [0, 2\pi)$ are independently, identically and uniformly distributed over $[0, 2\pi)$. So the expectation of $\cos(\beta_i)$ is

$$\frac{1}{2\pi}\int_{\beta=0}^{2\pi} \cos\beta \,\mathrm{d}\beta = 0.$$

And, the expected value of the sum is

$$\mathrm{E}\left[|H_m|^2\right] = m + \sum_{i=1}^{n}\sum_{\substack{k=1,\\ i\neq k}}^{n} \underbrace{\mathrm{E}\left[\cos\beta_i-\beta_k\right]}_{0} = m.$$

The root mean square of $h$ is therefore

$$|H_m|_{\text{rms}} = \sqrt{m}.$$

For the expectation there is no closed form known. Notice that even for small number of hops the analysis is complex, i.e. for $m \leq 4$, see [22]. A good approximation has been presented in 1905 by Lord Rayleigh [23] with the the probability distribution $\frac{2x}{m}e^{-x^2/m}$ for large $m$. The expectation of this approximation is $\frac{1}{2}\sqrt{\pi}\sqrt{m}$. Using the local central limit theorem leads to Proposition 2.1.2 (2.7) in [24]:

$$\mathrm{P}\left[|H_m| \geq s\sqrt{m}\right] \leq c \cdot e^{-\beta s^2}$$

in accordance with Rayleigh's approximation for some positive constant $c$ and $\beta$. So, $E(|H_m|) = \mathcal{O}(\sqrt{m})$ follows from this proposition.

If $\ell$ is not a multiple of $2\pi$, note that $\mathrm{E}\left[|H_{m,\ell}|\right]$ is possibly non-zero. We observe for $\beta_i \in [-\underline{\ell}, \underline{\ell}]$ for $\underline{\ell} = 2\pi\lfloor \ell/(2\pi)\rfloor$ the expectation above. The other case happens with probability $(\ell-\underline{\ell})/\ell \leq 2\pi/\ell$. So, the expected value of $|\beta_i|$ is at most $2\pi/\ell$. So, the overall expected number of $|H_{\ell,m}|$ is bounded by $2\pi m/\ell$. For $\ell > 2\pi\sqrt{m}$ we have

$$\mathrm{E}\left[|H_{\ell,m}|\right] \leq \sqrt{m}.$$

Therefore the standard deviation for general $\ell$ remains $\mathcal{O}(\sqrt{m})$ since the random variables $\beta$ are independent. ■

Clearly the distribution of phases differs from the uniform distribution. However, the simulations of the next chapter give some evidence that this behavior also holds for the correct distribution. For $\ell \geq 2\pi\sqrt{m}$ and $\ell \approx \frac{2\pi}{\lambda}\alpha d$ (for small enough $\alpha$) we get $\alpha \geq \frac{\lambda}{d}\sqrt{m}$ as the minimum angle for the random noise area. This bounds the side beams in our model to be within an angle range $[-\gamma, \gamma]$ with $\gamma \approx \frac{\lambda}{d}\sqrt{m}$.

We assume a unit power of 1 for all antennas and each antenna gets the same fraction $\frac{1}{m}$ of that overall power. Then the overall noise of the sender at a random angle outside the main and side beam area is

$$h_{\text{send}} = \frac{1}{m}\cdot\sqrt{m} = \frac{1}{\sqrt{m}}.$$

To this point we have only considered directed sending. When we also consider that the receiver has directed reception with multiple antennas in a random direction the overall attenuation is

$$h = \underbrace{\frac{1}{\sqrt{m}}}_{\text{sender}}\cdot\underbrace{\frac{1}{\sqrt{m}}}_{\text{receiver}} = \frac{1}{m}.$$

**Conjecture 1** *In SIMO (Single Input Multiple Output) the expected noise produced by a sender with a single antenna is $\mathcal{O}(1/\sqrt{m})$ when the receiver has $m$ multiple antennas randomly placed in the plane on a disk with diameter $d$ and the receiving angle is at random.*

**Conjecture 2** *In MISO (Multiple Input Single Output) the expected noise produced by a sender with $m$ multiple antennas randomly placed in the plane on a disk with diameter $d$ is $\mathcal{O}(1/\sqrt{m})$ at a receiver with a single antenna when the sending angle is at random.*

**Conjecture 3** *In MIMO (Multiple Input Multiple Output) the expected noise produced by a sender with $m_s$ multiple antennas randomly placed in the plane on a disk with diameter $d$ is $\mathcal{O}\left(1/\left(\sqrt{m_s}\cdot\sqrt{m_r}\right)\right)$ at a receiver with $m_r$ multiple antennas randomly placed in the plane on a disk with diameter $d$ when sending respectively receiving angle are at random. If sender and receiver antennas are homogenous with $m = m_r = m_s$ the noise is $\mathcal{O}(1/m)$.*

## V. Simulation

In this chapter we present numerical simulations to support the estimations presented in the analysis. Figure 6 shows the angle-dependent signal strength of a multiple antenna array where angle $\alpha = 0$ is the sending direction for directed communication. The signal strengths are normalized with maximum value 1 and computed for infinite distant targets to obtain our assumptions without an error. We compute

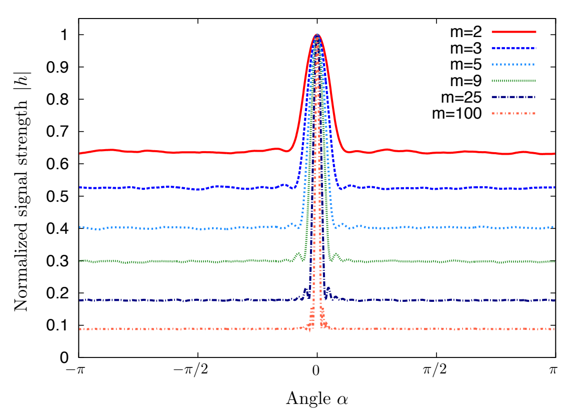



Fig. 6. The graph shows the normalized angular signal strength $|h|$ in the plane for different number of antennas $m$. Angle $\alpha = 0$ is always the target direction. The disc diameter for antenna placement is $d = 2\lambda\sqrt{m}$.

the radiation pattern for different numbers of antennas $m \in$

$\{2, 4, 6, 8, 10, 100, 1000\}$. The diameter of the antenna arrays is $d = 2\lambda\sqrt{m}$. For each number of antennas we average over $10,000$ simulations with a random positioned antenna array and a random transmission direction. The random noise is $1/\sqrt{m}$ as expected. The angle range of the main beam around $\alpha = 0$ decreases with increasing number of antennas. Especially for a high number of antennas one can spot two major side beams around the main beam. The distances of the side beams to the transmission direction $\alpha = 0$ also decrease with increasing number of antennas because of an increasing disc diameter $d$.

In the next experiment in Figure 7 we keep the disc area constant with a radius of $d = 2\lambda$ instead of increasing the area with the number of antennas $m$. The result shows that

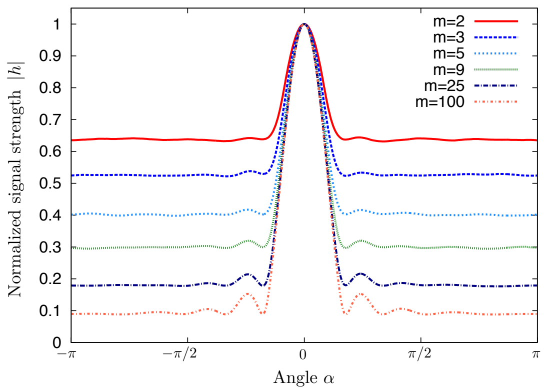



Fig. 7. The graph shows the normalized angular signal strength $|h|$ in the plane for different number of antennas $m$. Angle $\alpha = 0$ is always the target direction. The disc diameter for antenna placement is $d = 2\lambda$.

the main and side beams have for all numbers of antennas $m$ the same angle range. But the average white Gaussian noise decreases with $1/\sqrt{m}$ and strength of the side beam increases with increasing number of antennas $m$.

In the simulation presented in Figure 8 we keep the number of antennas constant to $m = 9$ and vary the disc area by increasing the disc diameter $d = k \cdot 2\lambda \cdot \sqrt{m}$ with a constant factor $k \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. The average white Gaussian noise keeps the same for all chosen $k$ but the angle range of the main beam and side beams decreases with increasing disc diameter $d$.

The signal strength in target direction $\alpha = 0$ is set up to be maximum and thus has variance 0 for random antenna placements under the given assumptions. We analyze the variance of the signal strength in the range of average Gaussian noise with angle $\alpha = \pi$ in Figure 9. We test the signal strength $h$ for the number of antennas $m \in [2, 1000]$ and $10,000$ random antenna placements each. The blue colored function is the standard deviance of the normalized signal strength $h$ multiplied with $\sqrt{m}$ and turns out to be constant with the conclusion of an average white Gaussian noise in the order of $1/\sqrt{m}$ as expected.

Only considering angular transmission simplifies indeed the network model since we only need to know the direction of

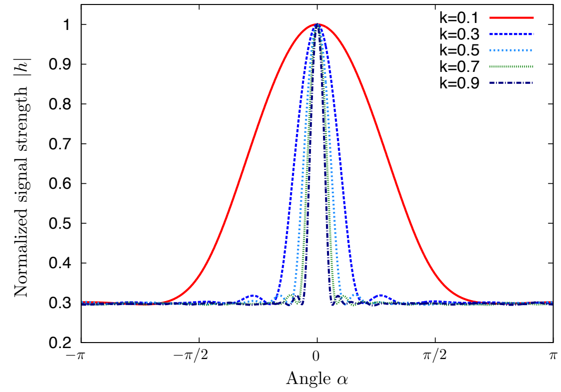



Fig. 8. The graph shows the normalized angular signal strength $|h|$ in the plane. Angle $\alpha = 0$ is always the target direction. The number of antennas is $m = 9$. The disc for antenna placement has diameter $d = 2k \cdot \sqrt{m} \cdot \lambda$.

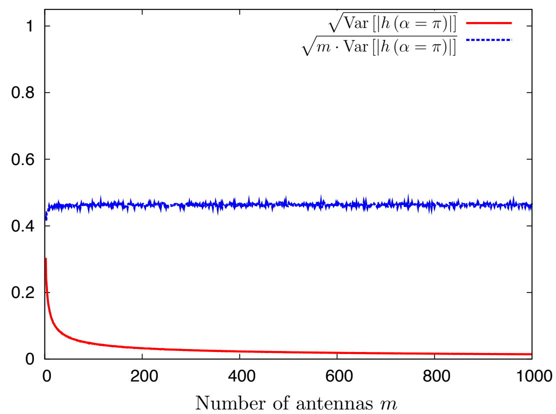
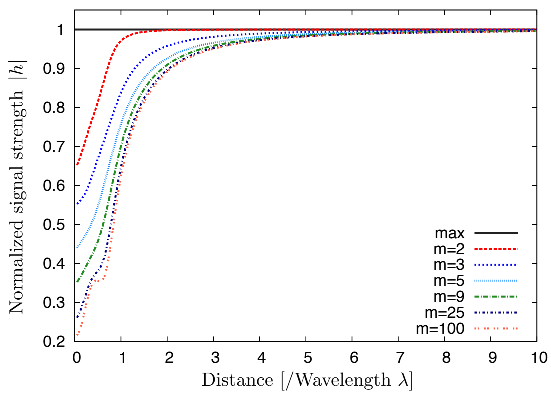



Fig. 9. The graph shows the standard deviation $\sqrt{\mathrm{Var}\left[|h|\right]}$ of the normalized signal strength $|h|$ at angle $\alpha = \pi$ when $\alpha = 0$ is the target direction. We estimate the average white Gaussian noise to be in order of $1/\sqrt{m}$.

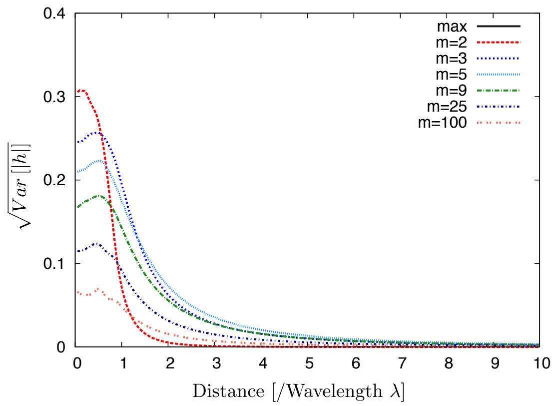
the target and not the actual location with additional distance information. But neglecting the distance leads to a small angle error resulting in not completely synchronized phases in the main beam. Figure 10 shows the simulation results for the signal strength in the main beam depending on the distance between the multiple antenna array and the target location. This is the result likewise for MISO and SIMO with communication between $m$ multiple antennas with a single antenna. The $m$ multiple antennas are randomly placed on a disc with diameter $d = 2\lambda$ with the wavelength $\lambda$. The distance between the array and the single antenna is measured from the centroid of the disc. The estimation error is maximum when placing the target position in the centroid of the disc of the antenna array with distance 0. Here, the signal strength $|h|$ decreases with $1/\sqrt{m}$ comparable to the average white Gaussian noise in a random direction. With increasing distance the error fades away and the signal strength converges to the maximum value possible 1.

## VI. Concluding Remarks

We present a multiple antenna model specializing on the beamforming capabilities of multiple antennas. In our setting




(a) Mean normalized signal strength $|h|$




(b) Standard deviation of signal strength $|h|$

Fig. 10. Optimizing the signal for a target direction and not considering the distance to the target causes an angle error. The simulation results show the impact on the signal strength $|h|$ depending on the distance for different antenna numbers $m$.

the $m$ antennas of a network node are randomly positioned on an area of $\Omega\left(m \cdot \lambda^2\right)$ where $\lambda$ is the wavelength of the carrier frequency. This achieves equally distributed phasing except for the main transmission beam. We estimate the width of the main transmission beam, where phases are highly correlated, with an angle in range $[-\kappa, \kappa]$ with $\kappa = \lambda/(2d)$ around the transmission angle. Beyond an angle of $\frac{\lambda}{d}\sqrt{m}$ the phasing is random and on the base of two-dimensional walk we estimate the expected noise to be attenuated by factor $1/m$ for MIMO and $1/\sqrt{m}$ for MISO respectively SIMO for $m$ antennas in comparison to the signal with maximum gain in transmission direction. Between the noise and the main beam we detect $\sqrt{m}$ side beams in the analytic estimation and simulations.

So, we classify the angles into three classes: the main beam is useful for transmission or reception; the side beams may cause interferences with other nodes; and the random noise range adds only little noise to the system. Another conclusion is that the beamforming capabilities of such MIMO systems can be improved by increasing the distance between the antennas or using frequencies with smaller wavelengths.

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