Online Multi-Robot Exploration of Grid Graphs with Rectangular Obstacles

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ABSTRACT

We consider the multi-robot exploration problem of an unknown $n \times n$ grid graph with oriented disjoint rectangular obstacles. All robots start at a given node and have to visit all nodes of the graph. The robots are unrestricted in their computational power and storage. In the local communication model the robots can exchange any information if they meet at the same node. In the global communication model all robots share the same knowledge.

In this paper we present the first nontrivial upper and lower bounds. We show that k robots can explore the graph using only local communication in time $O(n \log^2(n) + (f \log n)/k)$, where f is the number of free nodes in the graph. This establishes a competitive upper bound of $O(\log^2 n)$.

For the lower bound we show a competitive factor of $\Omega\left(\frac{\log k}{\log \log k}\right)$ for deterministic exploration and $\Omega\left(\frac{\sqrt{\log k}}{\log \log k}\right)$ for randomized exploration strategies using global communication.

Categories and Subject Descriptors

G.2.2 [**Mathematics of Computing**]: DISCRETE MATHEMATICSGraph Theory[Graph algorithms, Trees]; F.2.2 [**Theory of Computation**]: ANALYSIS OF ALGORITHMS AND PROBLEM COMPLEXITY Nonnumerical Algorithms and Problems

General Terms

Algorithms,Theory

Keywords

competitive analysis, mobile agent, robot, collective graph exploration

1. INTRODUCTION

Recent advancements allow the usage of robots in everyday use. Today's consumers are buying robotic vacuum cleaners and in the future one might expect the use of larger quantities of such robots. This alone is ample motivation for the investigation of exploration strategies of unknown terrains. Instead of an euclidean plain with scattered objects, we concentrate on the two-dimensional grid graph with oriented disjoint rectangular obstacles as a discrete model for this. The task is to explore all nodes of the graph. For this, each robot can only identify the current node and its neighboring nodes of the graph. In each step a robot can move to an adjacent node. Robots have unlimited memory, computational power and know their positions. In the local communication model they are able to exchange their findings with all the other robots if they meet in the same node. In the global communication model they can exchange their findings immediately.

We measure the efficiency of the robot exploration strategy by the competitive ratio, i.e. the number of steps needed by the robots in parallel following an exploration strategy on an unknown graph divided by the number of steps the robots need using an optimal exploration strategy (with the knowledge of the graph). For the deterministic lower bound we consider an adversary who chooses the graph knowing the exploration algorithm. To lower bound randomized strategies the adversary chooses a graph without knowing the random guesses of the randomized strategy.

Related Work.

For the exploration problem many different variants exist differing in reliability of sensor data, the number of robots, computational power, memory, range of communication, different kinds of graphs and computational or energy limitations (for a survey we refer to [15]). Most work on exploration of undirected graphs only handles the single robot case.

Competitive analysis of the exploration problem has been done by Dessmark et al. in [8] with a single robot. Dessmark also distinguishes between anchored and unanchored maps reducing the maps' effectiveness.

The hardness of the multi-robot exploration problem for trees is shown in [11]. They prove a lower bound for deterministic strategies with a competitive factor of $2 + \frac{1}{k}$. Later this bound was improved in [9], where a special graph, called Jellyfish-Tree has led to a lower bound of $\Omega(\frac{\log k}{\log \log k})$ for the competitive factor of a deterministic online exploration algorithm. Our work is based on this approach. As a side result we can generalize this result for randomized algorithms in this paper. Fraigniaud et al. prove in [11] the best positive result for competitive exploration of a tree with a competitive factor of $O\left(\frac{k}{\log k}\right)$.

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The grid model with oriented disjoint rectangular obstacles has been investigated for the online-shortest path problem [14] by Papadimitriou et al. Such graphs have been considered for a variant of the shortest path problem, where a robot tries to reach a line. They prove a tight upper and lower bound of a competitive factor of $O(\sqrt{n})$ [14]. For randomized strategies this bound can be improved to $O(n^{4/9} \log n)$ [6]. It is shown in [3] that a robot can move to a given point in time $O(n \log n)$ if the point is within a $n \times n$-grid with such obstacles. We will use this result to give the first polylogarithmic upper bound for multi-robot exploration.

Practical work has been done by Franchi et al. [12] using simulations to validate their algorithms. There is also a vast number of results for multi-robot exploration using real robots, e.g. a frontier based exploration by Yamauchi [16].

For a geometrical setting Albers et al. provide in [2] a lower bound on competitiveness. They allow one exploration robot to have unlimited optical vision and call the single robot exploration successful if the whole area has been seen by the robot. For directed graphs competitive analyses of single robot exploration have been presented by Albers et al. [1], Fleischer et al. [10] and Papadimitriou et al. [7]. Their results indicate that this is a harder problem than the exploration in undirected graphs. For unlabeled graphs exploration is unsolvable with a single robot but can be solved with a single pebble or another robot [4, 5]. Exploration of undirected graphs as considered here, is a variant of the online k-TSP and closely related to the exploration of a landscape.

Our Contribution.

In the next section we present the first multi-robot exploration algorithm for oriented rectangular disjoint obstacles in a $n \times n$-grid with a polylogarithmic competitive ratio. In particular we prove that k robots can explore such a grid in time $O(n \log^2 n + f/k \log n)$ where f is the number of (non-obstacle) nodes in the grid. Since the optimal strategy needs at least $\max\{2n-1, f/k\}$ steps we show a competitive bound of $O(\log^2 n)$. The algorithm is a divide-and-conquer algorithm which uses the single-robot-exploration strategy of [3].

In Section 3 we present the first nontrivial lower bound for deterministic multi-robot exploration for such grids. We show a lower bound for the competitive factor of $\Omega\left(\frac{\log k}{\log \log k}\right)$. The construction is based on the lower bound for multi-robot-exploration in trees presented in [9].

Then, in Section 4 we consider lower bounds for randomized exploration strategies and show a lower bound of $\Omega\left(\frac{\sqrt{\log k}}{\log \log k}\right)$ for grid graphs with rectangular disjoint obstacles. To prove this result, we first prove a new lower bound for randomized exploration strategies in trees of $\Omega\left(\frac{\log k}{\log \log k}\right)$.

2. EFFICIENT MULTI-ROBOT EXPLORATION

In [3] it is shown that in a $n \times m$-grid ($n \geq m$) with unknown oriented rectangular disjoint obstacles a robot can navigate to any point in time $O(n \log m)$ or to the obstacle in which the point lies. We use this result for a divide-and-conquer strategy. We use the following notations.

Let N, E, S, W denote the directions. A $\delta_1\delta_2$-path is a directed path which consists only of steps with directions δ_1 and δ_2. For neighbored directions $\delta_1, \delta_2 \in \{N, E, S, W\}$ a *greedy $\delta_1\delta_2$-path* is a path without obstacles where from the starting point the path goes to direction δ_1. Each time an obstacle occurs, the path takes a turn in direction δ_2 and continues until the way is free again in direction δ_1, then it continues in direction δ_1. From every point in the grid every greedy $\delta_1\delta_2$-path exists and has a maximum length of $2n-1$.

We need the notion of *surroundable regions*.

Definition 1. A *surroundable region* is a set of connected nodes which has a bounding path which is described by the concatenation of a greedy NW, WS, SE, and EN-path. Note that the complete $n \times m$-grid is such a surroundable region.

The continuous area $A(R)$ of a region R is the area of the region of R where the path and the region is interpreted geometrically.

Every surroundable region has a bordering path with a length of at most $4n-2$. For our divide-and-conquer algorithm we successively partition such regions. We also consider a geometric version of the grid graph in the Euclidean plane bounded to $[0, n]^2$. Obstacles are obviously modeled by rectangles. For the paths of the robot we consider series of line segments connecting the middle points of the empty squares representing the nodes of the graph.

LEMMA 2.1. *There exists a point p such that*

1. $Q_1(p) = Q_3(p)$ *and* $Q_2(p) = Q_4(p)$
2. $Q_1(p) + Q_3(p) \geq A(R)/2$ *or* $Q_2(p) + Q_4(p) \geq A(R)/2$.

PROOF. Consider the function $f_{13}(x, y) = Q_1(x, y) - Q_3(x, y)$. If x is smaller than any x-coordinate of a point in R and y is smaller than any y-coordinate of a point in R then $f_{13}(p_x, p_y) = A(R)$. If x is larger than any x-coordinate of a point in R and y is larger than any y-coordinate of a point in R then $f_{13}(p_x, p_y) = -A(R)$. Further, the function is continuous and decreases with x and y. Therefore, for each x there exists a y such that $f_{13}(x, y) = 0$ and for all y there exists a x such that $f(x, y) = 0$.

For the function $f_{24}(x, y) = Q_2(x, y) - Q_4(x, y)$ we can deduce the equivalent observations.

Given a rectangle $(x_1, y_1), (x_2, y_2)$ where $f_{13}(x_1, y_1) \leq 0$, $f_{13}(x_2, y_2) \geq 0$, $f_{24}(x_2, y_1) \leq 0$ and $f_{24}(x_1, y_2) \geq 0$, we can conclude that in one of the four equal-sized sub-rectangles this condition is preserved. The choice of the rectangle depends on the signs of $f_{13}(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$ and $f_{24}(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$. This implies the existence of a point p where $f_{13}(p) = f_{24}(p) = 0$.

Since $f_{13}(p) = f_{24}(p)$ it follows $Q_1(p) = Q_3(p)$ and $Q_2(p) = Q_4(p)$. If $Q_1(p) + Q_3(p) \geq Q_2(p) + Q_4(p)$ then

$$2(Q_1(p) + Q_3(p)) \geq Q_1(p) + Q_3(p) + Q_2(p) + Q_4(p) = A(R) .$$

Otherwise we have $Q_1(p) + Q_3(p) < Q_2(p) + Q_4(p)$ and therefore

$$2(Q_2(p) + Q_4(p)) > Q_1(p) + Q_3(p) + Q_2(p) + Q_4(p) = A(R) .$$

□

LEMMA 2.2. *Each surroundable region R can be partitioned into two surroundable regions R_1, R_2 such that $R_1 \cup R_2 = R$ and $A(R_i) \leq \frac{3}{4}A(R)$ for $i \in \{1, 2\}$. This can be done in time $O(n \log n)$ with a single robot.*

PROOF. For a fixed region R consider a point $p = (p_x, p_y)$ in this Euclidean space. Then we define $Q_1(p)$ as the area of R in the NE-quadrant (including obstacles). Similarly, we define Q_2, Q_3, Q_4 as the areas of R in the NW, SW and SE-quadrant. Clearly, the sum of all $Q_i(R)$ equals the area $A(R)$ of R. As visualisation we refer to Fig. 6 in the appendix.

This point p can be efficiently computed, since the region R is defined by horizontal boundaries. Now we navigate a robot to this

point p using the algorithm of [3]. If it lies within an obstacle, the algorithm will circle the obstacle, otherwise it will reach the node which is nearest to the point.

Assume that $Q_1(p)+Q_3(p) \geq A(R)/2$. Then, we will construct a path within the second and forth quadrant which divides R into R_1 and R_2. For this, we simply follow a greedy NW-path until we reach the boundary of R. Then, we follow a greedy SE-path starting from p until we reach the boundary of R. This path will not leave the second and forth quadrant and the sub-regions are again surroundable.

If p lies within an obstacle, we take the obstacle corner points in the second and forth quadrant with respect of p. Then, we construct NW-paths and SE-paths from these two points and combine them with two surrounding paths of the rectangular obstacles around p.

In both cases we have $A(R_1) \geq Q_1(p) \geq \frac{1}{4}A(R)$ and also $A(R_2) \geq \frac{1}{4}A(R)$.

If $Q_1(p) + Q_3(p) < A(R)/2$ then we have $Q_2(p) + Q_4(p) \geq A(R)/2$ and we make the symmetric construction within the first and third quadrant of p using greedy NE and SW-paths. Again, we get $A(R_1) \geq \frac{1}{4}A(R)$ and $A(R_2) \geq \frac{1}{4}A(R)$. Since $A(R_1) + A(R_2) = A(R)$ the claim follows. □

Algorithm 1 uses this partitioning to explore the square.

Algorithm 1 $O(\log^2 n)$-competitive multi-robot exploration of the $n \times n$ grid with k robots

```
1: Start with the full square as a single surroundable region
2: All robots start in the upper left corner
3: for i ← 1, 2, ..., log k do
4:     Partition all 2^(i-1) regions in parallel using one robot per region
5: end for
6: while Unexplored regions exist do
7:     Explore all k regions with one robot each using depth-first-search
8:     If a robot finishes the DFS it returns to the upper left corner
9:     if at least k/2 robots have returned to the upper left corner then
10:        Stop the entire exploration
11:        Partition all k/2 unexplored regions
12:    end if
13: end while
```

THEOREM 2.3. *Algorithm 1 can explore the $n \times n$ grid with k robots in time $O(n \log^2 n + n \log n \log k + (f \log n)/k)$ where f is the number of nodes in the grid (without obstacles) using the global communication model.*

PROOF. First note that a single robot can explore a connected area with f (non-obstacle) nodes using depth-first-search in time $2f$, but such an area cannot be explored with less than f steps by a single robot.

It takes at most $2\log_{4/3} n$ rounds of re-partitioning until all surroundable regions have size of at most 1, since the size of a region is reduced by at least a factor of $3/4$. Each partitioning takes $O(n \log n)$ steps for one robot. Moving to the left upper corner takes $2n-1$ steps using a greedy NE-path. So, the time of the lines 3–5 can be estimated by $O(n \log n \log k)$ steps. The while-loop (line 6) is executed at most $O(\log n)$ times. All partitioning steps in line 11 take therefore $O(n \log^2 n)$ steps.

For the exploration time we consider the rounds of the while-loop (lines 6–13). Let f_j denote the number of unexplored (free) cells at the beginning of the j-th round. There are two cases.

In the first case the loop finishes in round j since less than $k/2$ robots have returned and all regions are explored. So, more than $k/2$ robots have explored f_j free cells in parallel. This has taken at most $4f_j/k$ steps, since $k/2$ robots have explored at most f_j cells in parallel with DFS. So, the time for this round can be estimated by $4f/k$.

In the second case the $k/2$ robots have returned in the j-th round, but $k/2$ unexplored regions will be again completely revisited in the round $j+1$. Let R be an explored region with the largest number of free cells (given by F). All explored regions have been explored in at most $2F(R)$ steps. However, all $k/2$ unexplored regions must have had at least $\frac{1}{2}F(R)$ free cells since otherwise they would have been explored in this round by the DFS. Summarizing over the $k/2$ unexplored regions we have $\frac{1}{4}kF(R) \leq f$ and therefore $F(R) \leq 4f/k$ which results in an upper time bound for the exploration of $8f/k$ steps in this round.

Since there are at most $O(\log n)$ rounds in the while-loop there are at most $O((f \log n)/k)$ steps for the exploration. □

The global communication model can be replaced with a local communication scheme, if all robots stop the algorithm every $8n$ steps, move to the left upper node, communicate, and then return to their work. This needs $4n$ steps and increases the run-time by a constant factor.

COROLLARY 2.4. *Using the local communication model the $n \times n$ grid can be explored with k robots in time $O(n \log^2 n + n \log n \log k + (f \log n)/k)$ where f is the number of nodes in the grid (without obstacles) using the global communication model.*

Every optimal exploration strategy where all k robots start in the left upper corner needs at least $2n-1$ steps to reach the opposite corner. The other lower bound of f/k results from the optimal parallelization of the exploration of the f cells. Further note, that n^2 robots can explore the $n \times n$ grid in time $O(n \log n)$. For this, each robot navigates to its assigned node. This establishes a competitive factor of $O(\log n)$. For $k \leq n^2$ we have $\log k \leq 2 \log n$ and thus a run-time of $O(n \log^2 n + f(\log n)/k)$ compared to lower bound of $\Omega(n + f/k)$ resulting in the following corollary.

COROLLARY 2.5. *There is an exploration strategy to explore an $n \times n$-grid with oriented disjoint rectangular obstacles with a competitive run-time ratio of $O(\log^2 n)$ in the local communication model.*

3. LOWER BOUND FOR CONVEX GRID-GRAPH

Since we are interested in asymptotic behavior we assume the edge length n of the overall grid and the number of robots k to be powers of 2. For our construction we choose $n = k^2$. Our construction is inspired by the lower bound construction in [9], called the Jellyfish-Tree in Fig. 4. Our grid graph separates k rectangular grids, called poison areas, such that the paths between these areas have a length of at least $n/2$ and at most $O(n)$ (non obstacle) cells are located outside of these poison areas, see Fig. 1.

Given the paths of robots in a poison area visiting at most w nodes in total and each robot starting (or departing) at one of the four corners of an area of $n/2 \times n/k$ we construct the grid poison in the following way: For each $j \in \{1, \ldots, \log n - \log k\}$ consider

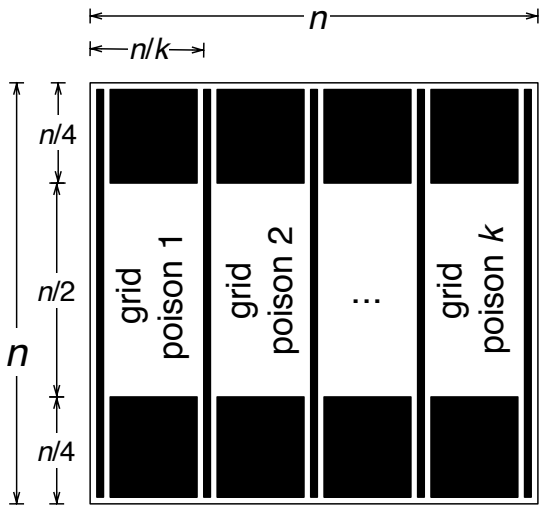

Figure 1: Separation of the grid poison areas

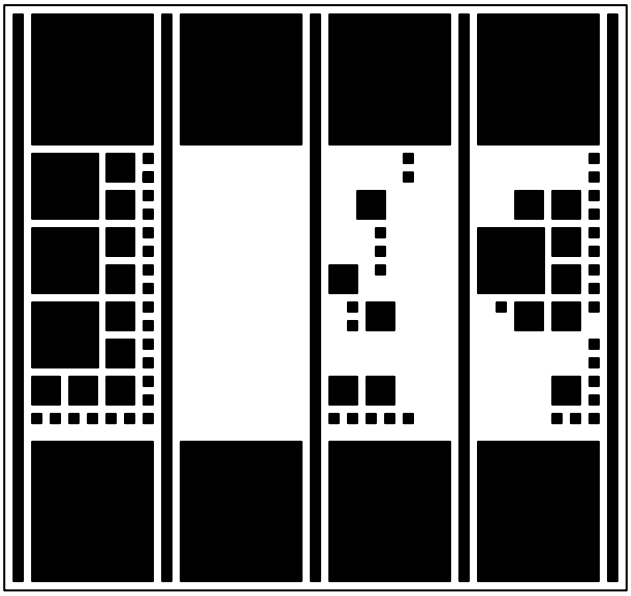

Figure 3: Example graph for the deterministic lower bound

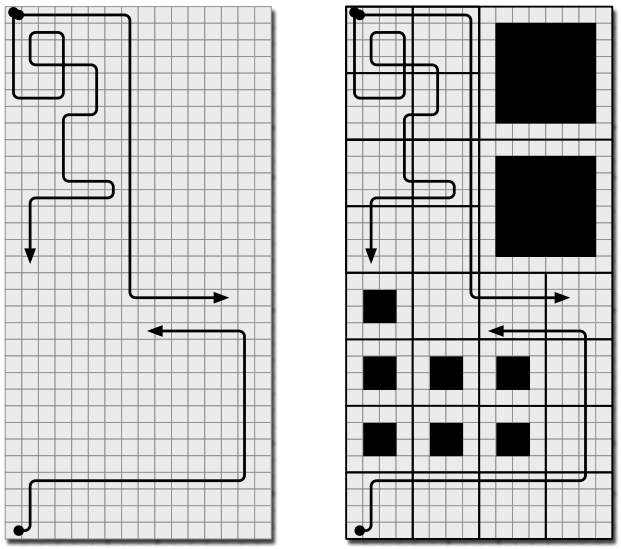

Figure 2: Deterministic construction of the grid poison based on the exploration paths of robots

a sub-grid of $2^j \times 2^j$-squares. If no robot has visited a square, then all cell nodes of this square but the border nodes (leftmost and rightmost column, lowermost uppermost row) will be removed. The border nodes are necessary to ensure that the square shaped obstacles remain disjoint and that the robot in the neighboring square does not learn anything. We call w the fooling size of the grid poison.

LEMMA 3.1. *Given a deterministic strategy of robots where the number of all traversed nodes of the robots is at most w in an $n \times m$ (with $n \geq m$) rectangle then the corresponding poison has at most size $O(n + w \log m)$. No visited cell is adjacent to a rectangle.*

PROOF. Clearly, the w traversed cells remain in the graph. When a subcell of dimension $2^j \times 2^j$ is removed, then $2^{j+2} - 2$ cells remain in the graph.

We estimate the number of such sub-grids which can be reached by any robots. Four cells can be reached without any traversal since the robots may start at the corners. The explored subcells are connected since they result from a set of paths starting in the corners. So, at most $4 + w2^{2-j}$ subcells of dimension $2^j \times 2^j$ can be reached.

We want to count all $2^j \times 2^j$ subcells which are replaced by an obstacle. Each such subcell has a neighbor cell (horizontal, vertical or diagonal) which has been visited by a robot or $m = 2^j$. Otherwise, the super-ordinate $2^{j+1} \times 2^{j+1}$ subcell would have been replaced by an obstacle. So the number of replaced subcells is at most $8(4 + w2^{2-j})$. In each of the $m \times m$ sub-squares of the $n \times m$ grid poison with at most w_i robot paths we observe at most the following number of free cells.

$$\sum_{j=2}^{\log m} 8(4 + w2^{2-j}) \cdot (2^{j+2} - 2) \leq 64m + 128w \log m$$

Summing over all such n/m squares we get at most $64n + 128w \log m = O(n + w \log m)$ free cells. □

While w visited sub-cells are not enough to encounter any obstacle, we show that visiting $O(w \log m + n)$ cells suffice to visit all

cells. An offline strategy with at most $k \leq n$ robots can do so in time $O(\frac{w}{k} \log m + n)$.

LEMMA 3.2. *A grid poison with fooling size w in a $n \times m$ rectangle ($n \geq m$) can be explored by k robots in time $O(n + \frac{w}{k} \log m)$ in the offline setting.*

PROOF. Partition the $p \leq 64n + 128w \log m$ empty cells of the grid poison in $b_i \times m$ rectangles such that $\sum_{i=1}^{k} b_i = n$ and that the number of empty cells in each rectangle is at most $\frac{p}{k} + m$. Each of the k robot explores one such rectangle. It needs n steps to reach the rectangle. For exploring such a rectangle, a robot may have to take a detour into neighbored rectangles because an obstacle hinders the direct path. Such detours have at most $4m$ cells. Furthermore, paths inside the rectangle may be traversed at most twice. This leads to an upper bound for $2b_i + 4m$ for the exploration within the rectangle. To reach the rectangle at most n steps are necessary. This results in an exploration time of at most $n + 6m + 2\frac{p}{k} = O(n + \frac{w \log m}{k})$. □

The fooling size of the grid poisons is chosen according to the following distribution:

$$w_{\sigma(i)} = \left\lceil \frac{kn}{(\log k)^2} \cdot \frac{1}{i} \right\rceil$$

Where σ denotes a permutation depending on the deterministic exploration strategy.

LEMMA 3.3. *There is an offline strategy which explores this graph within $O(n)$ steps using k robots.*

PROOF. Remember that $n = k^2$ and $m = k$. Define $W := \sum_{i=1}^{k} w_{\sigma(i)} \log k$ and note that $W = O\left(k + \frac{kn}{\log k}\right)$. An offline exploration strategy sends one robot in each grid poison for time cn to explore the grid poison. After this round, it sends $\left\lfloor \frac{w_{\sigma(i)}}{W} k \right\rfloor$ robots in each unexplored grid poison for time cn as well.

All cells with fooling size of at most $n/\log k$ can be explored within the first round. If $w_{\sigma(i)} > \frac{W}{k} = O(1)$ then at least one robot will explore the grid poison after this round. This is the case for $i \leq c\frac{k}{\log k}$ for some constant $c > 0$. Exploring such a poison costs time linear in

$$n + \frac{w_{\sigma(i)}}{\left\lfloor k \frac{w_{\sigma(i)}}{W} \right\rfloor} \log m = O\left(n + \frac{W}{k} \log m\right) = O(n) \,.$$

□

THEOREM 3.4. *Any deterministic exploration strategy needs at least $\Omega\left(n \cdot \frac{\log k}{\log \log k}\right)$ steps to explore this graph with k robots.*

A proof is analogous to the lower bound in [9] and can be found in Appendix A.1.

This implies an online time for any deterministic algorithm of

$$\Omega\left(n \cdot \frac{\log k}{\log \log k}\right)$$

leading directly to a lower bound for the competitive ratio of

$$\Omega\left(\frac{\log k}{\log \log k}\right)$$

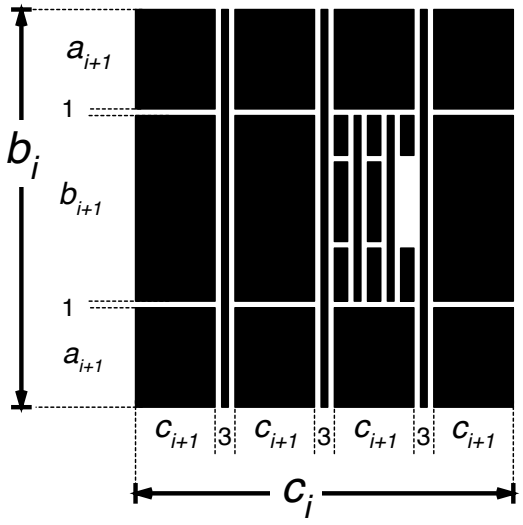

Figure 5: Recursive construction of poison areas for the randomized lower bound

4. LOWER BOUNDS FOR RANDOMIZED ALGORITHMS

Now we show that the lower bound given in [9] also applies for randomized algorithms. Consider the Jellyfish-Tree in Fig. 4. We use the same construction with k subtrees and a random permutation σ over $\{1, \ldots, k\}$. The i-th subtree consists of a path of length $t = k$ and a poison which is a tree of size $t \cdot s_{\sigma(i)}$ and depth $s_{\sigma(i)}$ where

$$s_{\sigma(i)} := \left\lceil \frac{k}{\log k} \cdot \frac{1}{i} \right\rceil$$

where in each level $k - 1$ leaves are connected to a parent and the graph continues at a random child, which we call the target child. The permutation σ is chosen uniformly at random. In [9] the following lemma has been shown regarding the offline exploration time.

LEMMA 4.1. *The Jellyfish-Tree can be explored in time $O(t)$ using k robots.*

Yao's principle [17] is used to show a lower bound for randomized strategies. We choose the randomized Jellyfish-Tree for a deterministic exploration strategy and show a lower bound on the expected time.

THEOREM 4.2. *For every randomized online exploration algorithm A, there is a graph such that the total time is at least $\Omega\left(\frac{\log k}{\log \log k}\right)$ times longer than the optimal time needed to explore this graph offline by k robots.*

Proof deferred to Appendix A.2.

4.1 Randomized Lower Bound for Grids

This theorem can be transferred to grids with rectangular bounds. Again we use a construction with poison areas like in the deterministic lower bound for the grid. It is clear that we cannot use the same construction, since it heavily depends on the knowledge of the deterministic strategy.

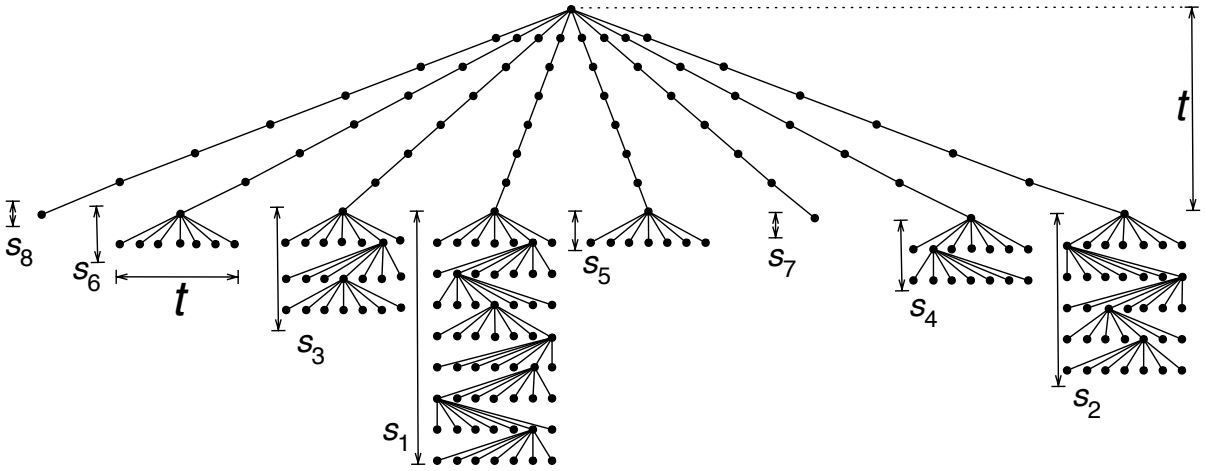

Figure 4: Jellyfish-Tree construction from [9].

Therefore, we use a randomized recursive construction where we place one poison within another. This "Matryoshka doll"-like construction has two features. First, it is hard to find the next enclosed grid poison. Second and most astonishingly, every next Matryoshka doll inside is twice as large as the outer one. This is possible, because of the fact that the outer poison has much fewer (yet longer) paths.

The outer grid construction is depicted in Fig. 2. We construct poison areas specifically designed for $k = \sqrt{n}$ randomized robots where the overall size of the grid is $n \times n$. The whole area is designed such that there is an offline strategy where k robots explore the graph in $O(n)$ steps.

We recursively define the poison areas as depicted in Fig. 5 starting with the uppermost layer which fits into the overall construction with $a_0 = \frac{1}{4}n - 2$, $b_0 = b = \frac{1}{2}n$ and $c_0 = c = \frac{n}{k+3}$ in the overall construction of Fig. 2. Further, define for $i \geq 0$

$$a_{i+1} = \frac{1}{4}b_i - 2\,, \quad b_{i+1} = \frac{1}{2}b_i\,, \quad c_{i+1} = c_i\frac{1}{2^{2i}}$$

So, we have the closed form for $i \geq 0$:

$$a_i = n\,2^{-i-2} - 2\,, \quad b_i = n\,2^{-i-1}\,, \quad c_i = \frac{n}{k2^{i(i+1)}}$$

This recursive definition ends when $c_{r+1} \leq 1$ for some r. Therefore

$$\frac{n}{k2^{(r+1)(r+2)}} \geq 1$$

which is implied by

$$\log n \geq \log k + (r+1)^2$$

And therefore:

$$r \leq \sqrt{\log n - \log k} - 1 = \frac{\sqrt{2}}{2}\sqrt{\log n} - 1$$

Note that the recursive constructions replace at most one of the inner rectangles with the next level. In this construction one of the inner obstacles is replaced by another element. In the lowest level this obstacle is a barrier. The following lemma describes the length of all paths in the fixed level i of the recursion.

LEMMA 4.3. *For a grid poison of level $i \geq 1$ the complete area to be explored is at most $n2^i$.*

PROOF. We have at least $\lceil \frac{c_i}{c_{i+1}+3} \rceil$ vertical paths of length b_i which have an overall length of $b_i = n2^{-i-1}$. Note that by definition

$$\frac{c_i}{c_{i+1}} = 2^{2i}$$

Therefore

$$\frac{c_i}{c_{i+1}+3}b_i \leq 2^{2i}\; n\; 2^{-i-1} = n2^{i-1}$$

The length of all horizontal paths is bounded by $4c_i \leq n$. □

Define the workload of an exploration strategy in a poison as the sum of all paths of all robots.

LEMMA 4.4. *For all $p \in [0,1]$ in a grid poison of level $i \geq 1$ the next recursive grid poison has not been found with a workload of at most $\frac{1}{2}pn2^i$ with probability $1-p$.*

PROOF. Consider the paths of a deterministic strategy of length $w = pn2^i$. The expected number of possible poisons that can be inspected with this workload is at most $\frac{1}{2}\frac{pn2^i}{b_i} = \frac{1}{2}\frac{pn2^i}{n2^{-i-1}} = p2^{2i}$. Clearly the probability is p for finding the correct target and therefore $1-p$ for failing to do so. □

Now we choose the levels ℓ_j of the poisons $1, \ldots, k$ according to the following distribution where σ is a random permutation over $\{1, \ldots, k\}$.

$$s_{\sigma(i)} = \frac{tk}{\log k} \cdot \frac{1}{i}$$

and

$$\ell_j = \lfloor \log s_j \rfloor$$

The maximum size of such a grid poison is bounded by $t2^{O(\sqrt{\log k})}$. So, we replace our distribution of poison sizes with

$$s_{\sigma(i)} = \begin{cases} \frac{tk}{\log k} \cdot \frac{1}{i} & \text{if } \frac{k}{2^{O(\sqrt{\log k})\log k}} \leq i \\ n2^{O(\sqrt{\log k})} & \text{else} \end{cases}$$

where we round to the next power of two.

THEOREM 4.5. *For every randomized online exploration algorithm there is a grid graph with disjoint rectangular obstacles such that the total time is at least* $\Omega\left(\frac{\sqrt{\log k}}{\log\log k}\right)$ *times longer than the optimal time needed to explore this graph offline by* k *robots.*

The full proof can be found in Appendix A.3. The proof is analogous to the proof of the randomized lower bound for trees. One main difference is that we do not prove with high probability, but with probability $1-\frac{1}{\log n}$. This probability for each poison is large enough since the expected number of unexplored poisons is considered.

Another main difference is that the number of rounds is now limited by $r = \frac{\sqrt{\log k}}{\log\log k}$. This is the reason for the worse lower bound.

The lower bounds can be easily generalized to robots with vision where each cell needs only to be seen by the robots (and not necessarily visited). This can be done by placing small view obstructing squares at all junctions in the lower bound construction presented here.

5. CONCLUSIONS

We consider multi-robot exploration where robots know their locations and are computationally well equipped. From the algorithmic perspective little is known. It turns out that even for trees the question how well an unknown graph can be explored is wide open between the lower bound for the competitive time ratio of $\Omega(\log k/\log\log k)$ and the upper bound of $O(k/\log k)$ for k robots. In this paper we have generalized the lower bound to random algorithms. But note that the trivial lower bound of 1 and the trivial upper bound of $O(k)$, where only one robot explores the tree, are not far from the known bounds.

Planar sceneries with disjoint oriented rectangular obstacles have been considered so far only for single robot exploration. We present an efficient online algorithm which explores such areas with time overhead of factor $O(\log^2 n)$ compared to the optimal solution in an $n \times n$ grid. As lower bounds we prove $\Omega(\log k/\log\log k)$ for deterministic strategies and $\Omega(\sqrt{\log k}/\log\log k)$ for random strategies with k robots. So, there is only a logarithmic gap between the upper and the lower bound.

For multi-robot exploration in general graphs little is known and finding efficient algorithms is a pressing research topic since the robotic exploration plays an increasing part in practical research.

6. REFERENCES

[1] S. Albers and M. R. Henzinger. Exploring Unknown Environments. *SIAM Journal on Computing*, 29(4):1164, 2000.

[2] S. Albers, K. Kursawe, and S. Schuierer. Exploring unknown environments with obstacles. In *Proceedings of the tenth annual ACM-SIAM symposium on Discrete algorithms*, SODA '99, pages 842–843, Philadelphia, PA, USA, 1999. Society for Industrial and Applied Mathematics.

[3] E. Bar-Eli, P. Berman, A. Fiat, and P. Yan. Online navigation in a room. *J. Algorithms*, 17:319–341, November 1994.

[4] M. A. Bender. The power of team exploration: Two robots can learn unlabeled directed graphs. In *In Proceedings of the Thirty Fifth Annual Symposium on Foundations of Computer Science*, pages 75–85, 1994.

[5] M. A. Bender, A. Fernández, D. Ron, A. Sahai, and S. Vadhan. The power of a pebble: Exploring and mapping directed graphs. *Information and Computation*, 176(1):1 – 21, 2002.

[6] P. Berman, A. Blum, A. Fiat, H. Karloff, A. Rosén, and M. Saks. Randomized robot navigation algorithms. In *Proceedings of the seventh annual ACM-SIAM symposium on Discrete algorithms*, SODA '96, pages 75–84, Philadelphia, PA, USA, 1996. Society for Industrial and Applied Mathematics.

[7] X. Deng and C. Papadimitriou. Exploring an unknown graph. In *Foundations of Computer Science, 1990. Proceedings., 31st Annual Symposium on*, pages 355 –361 vol. 1, oct 1990.

[8] A. Dessmark and A. Pelc. Optimal graph exploration without good maps. *Theor. Comput. Sci.*, 326:343–362, October 2004.

[9] M. Dynia, J. Lopuszanski, and C. Schindelhauer. Why robots need maps. In *Proceedings of the 14th international conference on Structural information and communication complexity*, SIROCCO'07, pages 41–50, Berlin, Heidelberg, 2007. Springer-Verlag.

[10] R. Fleischer and G. Trippen. Exploring an unknown graph efficiently. In G. Brodal and S. Leonardi, editors, *Algorithms – ESA 2005*, volume 3669 of *Lecture Notes in Computer Science*, pages 11–22. Springer Berlin / Heidelberg, 2005. 10.1007/11561071_4.

[11] P. Fraigniaud, L. Gąsieniec, D. R. Kowalski, and A. Pelc. Collective tree exploration. *Netw.*, 48:166–177, October 2006.

[12] A. Franchi, L. Freda, G. Oriolo, and M. Vendittelli. A randomized strategy for cooperative robot exploration. In *Robotics and Automation, 2007 IEEE International Conference on*, pages 768 –774, april 2007.

[13] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58:13–30, 1963.

[14] C. H. Papadimitriou and M. Yannakakis. Shortest paths without a map. *Theor. Comput. Sci.*, 84:127–150, July 1991.

[15] N. S. V. Rao, S. Kareti, W. Shi, and S. S. Iyengar. Robot navigation in unknown terrains: Introductory survey of non-heuristic algorithms. Technical Report ORNL/TM-12410:1–58, Oak Ridge National Laboratory, July 1993.

[16] B. Yamauchi. Frontier-based exploration using multiple robots. In *Proceedings of the second international conference on Autonomous agents*, AGENTS '98, pages 47–53, New York, NY, USA, 1998. ACM.

[17] A. C.-C. Yao. Probabilistic computations: Toward a unified measure of complexity. In *Foundations of Computer Science, 1977., 18th Annual Symposium on*, pages 222 –227, 31 1977-nov. 2 1977.

APPENDIX

A. OMITTED PROOFS

A.1 Proof for Lower Bound with Deterministic Exploration Strategy

PROOF. For the lower bound argument we consider rounds of length $n/2$. Note that in each round a robot can visit only one poison grid. Let $r_{t,i}$ denote the number of robots that visit poison i in round t. At the beginning of each round the adversary allows the robots to know the size of some of the poisons while a decreasing number of poisons remain of unknown size. As soon the robots learn the size of the poison the poison is lost and no more exploration costs are accounted for (since the overall offline exploration cost is $O(n)$).

Furthermore, we do not count the costs of replacement from one poison grid to another.

Let u_i be the number of unexplored poisons after the i-th round. Then, at least $\frac{u_i}{2}$ poisons are explored by at most $2\frac{k}{u_i}$ robots. By our construction we can ensure that

$$u_i \geq \frac{k}{2^i \log^{2i} k} .$$

For this we choose the permutation σ in the following way. All the poisons are sorted according to whether more than $2\frac{k}{u_i}$ robots visit a poison in the i-th round. For this we place the poison grids with larger number of robots in the first round at the beginning, then we continue within the set at the beginning by sorting poison grids according to the robots of the second round and so forth.

By induction before round i at least u_{i-1} poisons are unexplored. Now the search algorithm can place at most k robots among those poisons and at least $\frac{u_{i-1}}{2}$ poisons are explored by at most $2\frac{k}{u_{i-1}}$ robots in round i. So, the robots cannot explore those poisons where

$$s_{\sigma(i)} \geq n2^i \log^{2i-2} k \geq kn \left(\frac{1}{u_{i-1}} + \frac{1}{u_{i-2}} + \ldots + \frac{1}{u_0} \right)$$

(by induction) since the robots have only time $n/2$ to explore the poison area. The number of poisons u_i of this size can be evaluated by using the definition of the distribution.

$$\frac{kn}{u_i (\log k)^2} \leq n2^i \log^{2i-2} k$$

So, we have

$$u_i \geq \frac{k}{2^i \log^{2i} k}$$

which proves the number of unexplored poisons by induction.

Note that for $r = \frac{1}{4} \frac{\log k}{\log \log k}$ we have

$$u_r = \frac{k}{2^{2r} \log^{2r} k} \geq \frac{k}{k^{\frac{1}{2 \log \log k}} 2^{\frac{1}{2} \log k}} \geq \frac{k}{k^{\frac{1}{2}} k^{\frac{1}{2}}} = 1 .$$

By this construction we have at least $\Omega\left(\frac{\log k}{\log \log k}\right)$ rounds with unexplored poison grids where each of the rounds have a run-time of $\frac{n}{2}$. □

A.2 Proof for Lower Bounds for Randomized Algorithms

PROOF. We consider rounds of length t. In each round a robot can visit only one poison. The deterministic exploration strategy knows the graph family. Therefore it has determined a poison if it has found all target children.

Now in each round of length t steps a different numbers of robots might explore a poison.

LEMMA A.1. *The probability that in a round a target child in depth ℓ is explored with $k' \leq t$ robots in less time than $\frac{1}{2}\frac{t}{k'}$ is at most $e^{-\frac{1}{8}\ell}$.*

PROOF. We assume that all k' robots test different children where each child is a target child with equal probability. Then, the probability to find the target child of the next level within i steps is $i \cdot \frac{k'}{t}$.

Define the random variable X which denotes the number of steps to find one target child with k' robots. Then $P[X = j] = \frac{k'}{t}$ for $j \in \{1, \ldots, \lceil t/k' \rceil - 1\}$ and $P[X = \lceil t/k' \rceil] = \frac{t - k' \bmod t}{t}$. Clearly, $\frac{1}{2}\lfloor t/k' \rfloor \leq E[X] \leq \frac{1}{2} t/k'$.

If $k' \geq t/2$ the target child may be found in each step. Otherwise if $k' \leq t_2$ we can bound the number of steps to find a series of ℓ target children using Hoeffding's tail inequality. Assume ℓ independent target children and let $S_\ell = \sum_{j=1}^{\ell} X_{j,i}$, where $X_{j,i}$ denotes the random variable above for k' robots. Then by the tail inequality in [13] we have for all $t \geq 0$ and $a_i = 1$ and $b_i = \lceil t/k' \rceil$

$$P[S_\ell - E[S_\ell] \leq -\delta] \leq e^{-2\delta^2 / \sum_{i=1}^{\ell} (b_i - a_i)^2}$$

Since $b_i - a_i \leq t/k'$ we have

$$P\left[S_\ell - E[S_\ell] \leq -\delta\right] \leq e^{-2\delta^2 k'^2/(\ell t^2)}$$

We choose $\delta = \frac{1}{2}E[S_\ell] \geq \frac{1}{2}\ell \cdot \lfloor \frac{t}{k'} \rfloor$ and get for $k' \leq t/2$

$$\begin{aligned} P\left[S_\ell \leq \frac{1}{2}E[S_\ell]\right] &\leq e^{-\frac{1}{2}\ell^2 \lfloor t/k' \rfloor^2 k'^2/(\ell t^2)} \\ &\leq e^{-\frac{1}{2}\ell(1-k'/t)^2} \leq e^{-\frac{1}{8}\ell} \end{aligned}$$

Remember that for $k' > t/2$ we have

$$P\left[S_\ell \leq \frac{1}{2}E[S_\ell]\right] = 0$$

The probability that a target child in depth ℓ is explored with k' robots in less time than $\frac{1}{2}\frac{t}{k'}$ is at most $e^{-\frac{1}{8}\ell}$. □

This implies the following corollary which shows that with high probability $1 - \frac{1}{n^2}$ that maximum speedup by randomization in a poison is a factor of $O(\log n)$.

COROLLARY A.2. *The probability that in a round of length t a target child in depth $16 \ln n$ is explored with $k' \leq t = k$ robots in less time than $\frac{1}{2}\frac{t}{k'}$ is at most $\frac{1}{n^2}$.*

So, k' robots in a round of length t can only find all target children in depth of at most $32k' \ln n = (64 \ln 2)k' \log k$ with probability $1 - \frac{1}{n^2}$.

Now in the first round we have k robots which are (deterministically) assigned to one poison each, but one necessarily each poisons receives the one robot. They have at most time t to explore each poison.

Consider poisons of depth of at least $c^j \log^j k$ for $c = 2^8$. There are $\frac{k}{c^j \log^{j+1} k}$ many such poisons. Such a poison cannot be explored with less than $\frac{c^j \log^{j-2} k}{2^6 \ln 2}$ robots with high probability. Then, only target children up to depth $c^j \log^{1-j} k$, i.e. a fraction of $\frac{1}{\log k}$ of the poison can be explored with high probability.

So, there at most $\frac{2^6 (\ln 2) k}{c^j \log^{j-2} k}$ poisons which may have enough robots and these poisons are randomly distributed over the set of all poisons. The probability that more than $\frac{c^j \log^{j-2} k}{2^6 \ln 2}$ robots are assigned to a poison is at most $\frac{2^6 (\ln 2)}{c^j \log^{j-2} k}$.

The expected number of explored poisons of this depth $c^j \log^j k$ after the first round is at most

$$r = \frac{k}{c^j \log^{j+1} k} \frac{2^6 (\ln 2)}{c^j \log^{j-2} k} = \frac{2^6 (\ln 2) k}{c^{2j} \log^{2j-1} k} .$$

We can apply a Chernoff bound since the explored poisons are negatively correlated: The exploration of a poison decreases the probability that another poison is explored. For $r \geq 8 \ln n$ we get with high probability that at most $2r$ poisons are explored with

high probability which is the case for $2 \le j \le \frac{1}{4}\log k/\log\log k$. Further, note that for $j \ge 2$

$$2r \le \frac{k}{c^{j+1}\log^{j+1} k}$$

since

$$\frac{2^7(\ln 2)k}{c^{2j}\log^{2j-1} k} \le \frac{k}{c^{j+1}\log^{j+1} k}$$

and

$$c^j \log^j k \ge (2^7 \ln 2) c \log^2 k\ .$$

because $c \ge 2^7 \ln 2$.

Hence the number of unexplored poisons of depth at least $c^j \log^j k$ for $j \ge 2$ is at least

$$\left(1-\frac{1}{c}\right)\frac{k}{c^j \log^{j+1} k}$$

after the first round with high probability.

By induction, at the beginning of the $(u+1)$-th round we have at least $(1-\frac{1}{c})^u k c^{-j}\log^{-j-1} k$ unexplored poisons of depth at least $c^j \log^j k$ for $j \ge 2u$ and $j \le \frac{1}{4}\log\log n/\log n$. We assume that these bounds are tight.

Again, for poisons of depth at least $c^j \log^j k$ a number of $\frac{c^j \log^{j-2} k}{2^6 \ln 2}$ of robots is not able to explore more than a fraction of $\frac{1}{\log k}$ of such poisons. So, there at most $\frac{2^6(\ln 2)k}{c^j \log^{j-2} k}$ poisons which may have enough robots and these poisons are randomly distributed over the set of all unexplored poisons which is at least $(1-\frac{1}{c})^u k c^{-2u}\log^{-2u-1} k$. The probability that enough robots are assigned to a poison is therefore at most

$$\frac{2^6(\ln 2)k}{c^j \log^{j-2} k}\left(\left(1-\frac{1}{c}\right)^u k c^{-2u}\log^{-2u-1} k\right)^{-1} = 2^6(\ln 2)c^{2u-j}\log^{2u-j+1} k$$

The expected number of explored poisons is at most

$$\begin{aligned} r &= \frac{\left(1-\frac{1}{c}\right)^u k}{c^j \log^{j+1} k}\cdot 2^6(\ln 2)c^{2u-j}\log^{2u-j+1} k \\ &= 2^6(\ln 2)\left(1-\frac{1}{c}\right)^u c^{2u-2j} k \log^{2u-2j-2} k \end{aligned}$$

Note that for $j \ge 2u+2$

$$2r \le \left(1-\frac{1}{c}\right)^u \frac{k}{c^{j+1}\log^{j+1} k}$$

since

$$2^7(\ln 2)\left(1-\frac{1}{c}\right)^u c^{2u-2j} k \log^{2u-2j-2} k \le \left(1-\frac{1}{c}\right)^u \frac{k}{c^{j+1}\log^{j+1} k}$$

and

$$c^{j-2u}\log^{j-2u} k \ge (2^7 \ln 2) c \log^2 k\ .$$

because $c \ge 2^7 \ln 2$. Again we can apply Hoeffding's bound since the explored poisons are negatively correlated. Applying Hoeffding's bound for $r \le 8\ln n$ we get with high probability that at most $2r$ poisons are explored with high probability which is the case for $2u \le j \le \frac{1}{4}\log k/\log\log k$.

So, the number of unexplored poisons of depth at least $c^j \log^j k$ for $j \ge 2u+2$ is at least

$$\left(1-\frac{1}{c}\right)\left(1-\frac{1}{c}\right)^u \frac{k}{c^j \log^{j+1} k}$$

after the $(u+1)$-th round with high probability, which proves the induction.

Since for all $u \le \frac{1}{4}\frac{\log k}{\log\log k}$ we can find unexplored poisons with high probability, the claim follows.

A.3 Proof for Randomized Lower Bound for Grids

PROOF. The proof is analogous to the proof of the randomized lower bound for graphs. The first difference is that we do not prove with high probability, but with probability $1-\frac{1}{\log n}$. This probability for each poison is large enough since the expected number of unexplored poisons is considered.

The second difference is that the number of rounds is now limited by $r = \frac{\sqrt{\log k}}{\log\log k}$. This is the reason for the worse lower bound. We consider rounds of length $n/2$. In each round a robot can visit only one poison grid. We use Yao's principle [17]and consider a deterministic strategy on the random graphs.

Consider poison grids of level of at least $\log(c^j \log^j k)$ for $c = 2^8$. There are $\frac{k}{c^j \log^{j+1} k}$ many such poison grids. Not even a fraction of $\frac{1}{\log^2 k}$ of such a poison grid can be explored in a round of length $n/2$ with less than $c^j \log^{j-2} k$ robots with probability $1-\frac{1}{\log^2 k}$. We can bound the number of poisons that are explored in the error case with Chernoff bounds. If $\frac{k}{c^j \log^{j-2} k} \ge 8\ln n$ then the error probability that more than $2\frac{k}{c^j \log^{j-2} k}$ such poison grids are explored is at most $\frac{1}{n^2}$.

So, there at most $3\frac{k}{c^j \log^{j-2} k}$ poison grids which may have enough robots and these poisons are randomly distributed over the set of all poisons w.h.p. The probability that more than $c^j \log^{j-2} k$ robots are assigned to a poison is at most $c^{-j}\log^{2-j} k$.

The expected number of explored poisons of this level $\log(c^j \log^j k)$ after the first round is at most

$$r = \frac{k}{c^j \log^{j+1} k}\frac{3}{c^j \log^{j-2} k} = \frac{3}{c^{2j}\log^{2j-1} k}$$

We can apply a Chernoff bound since the explored poisons are negatively correlated: The exploration of a poison decreases the probability that another poison is explored. For $r \ge 8\ln n$ we get with high probability that at most $2r$ poisons are explored with high probability which is the case for $2 \le j \le \sqrt{\log k}/\log\log k$.

Further, note that for $j \ge 2$

$$2r \le \frac{k}{c^{j+1}\log^{j+1} k}$$

since

$$\frac{6k}{c^{2j}\log^{2j-1} k} \le \frac{k}{c^{j+1}\log^{j+1} k}$$

and

$$c^j \log^j k \ge 6c\log^2 k\ .$$

if we choose $c \ge 6$.

Hence the number of unexplored poison grids of level at least $\log(c^j \log^j k)$ for $j \ge 2$ is at least

$$\left(1-\frac{1}{c}\right)\frac{k}{c^j \log^{j+1} k}$$

after the first round with high probability.

After each round the robots may be placed on different poison grids. Although the robots need time $n/2$ to travel from one poison grid to another we do not use this feature, since we also have to deal with robots which do not travel to new poisons which complicates the analysis. It is easy to see that taking the travel time into account accounts only for a constant factor.

By induction, at the beginning of the $(u+1)$-th round we have at least $(1-\frac{1}{c})^u kc^{-j}\log^{-j-1}k$ unexplored poison grids of level at least $\log(c^j\log^j k)$ for $j \geq 2u$ and $j \leq \sqrt{\log\log n}/\log n$. We assume that these bounds are tight, i.e. we allow the robot strategy learn about the situation in the other poison grids.

Again, for poison grids of level at least $\log(c^j\log^j k)$ a number of $c^j\log^{j-2}k$ of robots is not able to explore more than a fraction of $\frac{1}{\log^2 k}$ of such poisons with probability $1-\frac{1}{\log^2 k}$. Again we bound the error case by Chernoff bound if $\frac{k}{c^j\log^{j-2}k} \geq 8\ln n$ then the error probability that more than $2\frac{k}{c^j\log^{j-2}k}$ such poison grids are explored is at most $\frac{1}{n^2}$.

So, there at most $\frac{3k}{c^j\log^{j-2}k}$ poisons which may have enough robots and these poisons are randomly distributed over the set of all unexplored poisons which is at least $(1-\frac{1}{c})^u kc^{-2u}\log^{-2u-1}k$. The probability that enough robots are assigned to a poison is therefore at most

$$\frac{3}{c^j\log^{j-2}k}\left(\left(1-\frac{1}{c}\right)^u kc^{-2u}\log^{-2u-1}k\right)^{-1} = 3c^{2u-j}\log^{2u-j+1}k$$

The expected number of explored poisons is at most

$$\begin{aligned} r &= \frac{\left(1-\frac{1}{c}\right)^u k}{c^j\log^{j+1}k}\cdot 3c^{2u-j}\log^{2u-j+1}k \\ &= 3\left(1-\frac{1}{c}\right)^u c^{2u-2j}k\log^{2u-2j-2}k \end{aligned}$$

Note that for $j \geq 2u+2$

$$2r \leq \left(1-\frac{1}{c}\right)^u \frac{k}{c^{j+1}\log^{j+1}k}$$

since

$$6\left(1-\frac{1}{c}\right)^u c^{2u-2j}k\log^{2u-2j-2}k \leq \left(1-\frac{1}{c}\right)^u \frac{k}{c^{j+1}\log^{j+1}k}$$

and

$$c^{j-2u}\log^{j-2u}k \geq 6c\log^2 k\,.$$

because $c \geq 6$.

Again we can apply Hoeffding's bound since the explored poisons are negatively correlated. Applying Hoeffding's bound for $r \geq 8\ln n$ we get with high probability that at most $2r$ poisons are explored with high probability which is the case for $2u \leq j \leq \sqrt{\log k}/\log\log k$.

So, the number of unexplored poisons of depth at least $c^j\log^j k$ for $j \geq 2u+2$ is at least

$$\left(1-\frac{1}{c}\right)\left(1-\frac{1}{c}\right)^u \frac{k}{c^j\log^{j+1}k}$$

after the $(u+1)$-th round with high probability, which proves the induction.

So, for $\sqrt{\log n}/\log\log n$ rounds of length n there will be unexplored poisons with high probability. □

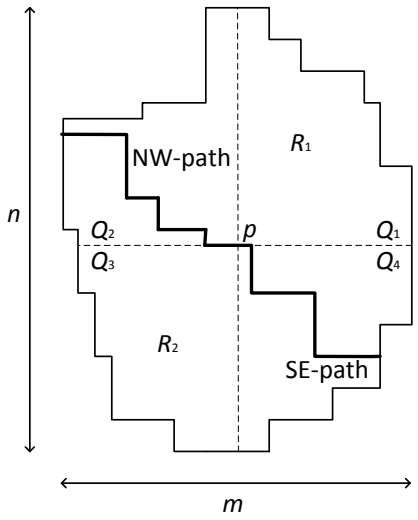

Figure 6: Partitioning an area for the efficient exploration