Broadcasting in Logarithmic Time for Ad Hoc Network Nodes on a Line Using MIMO

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ABSTRACT

We consider n wireless ad hoc network nodes with one antenna each and equidistantly placed on a line. The transmission power of each node is just large enough to reach its next neighbor. For this setting we show that a message can be broadcasted to all nodes in time $\mathcal{O}(\log n)$ without increasing each node's transmission power. Our algorithm needs $\mathcal{O}(\log n)$ messages and consumes a total energy which is only a constant factor larger than the standard approach where nodes sequentially transmit the broadcast message to their next neighbors. We obtain this by synchronizing the nodes on the fly and using MIMO (multiple input multiple output) techniques.

To achieve this goal we analyze the communication capacity of multiple antennas positioned on a line and use a communication model which is based on electromagnetic fields in free space. We extend existing communication models which either reflect only the sender power or neglect the locations by concentrating only on the channel matrix. Here, we compute the scalar channel matrix from the locations of the antennas and thereby only consider line-of-sight-communication without obstacles, reflections, diffractions or scattering.

First, we show that this communication model reduces to the SINR power model if the antennas are uncoordinated. We show that n coordinated antennas can send a signal which is n times more powerful than the sum of their transmission powers. Alternatively, the power can be reduced to an arbitrarily small polynomial with respect to the distance. For coordinated antennas we show how the well-known power gain for MISO (multiple input single output) and SIMO (single input multiple output) can be described in this model. Furthermore, we analyze the channel matrix and prove that in the free space model no diversity gain can be expected for MIMO.

Finally, we present the logarithmic time broadcast algorithm which takes advantage of the MISO power gain by self-coordinating wireless nodes.

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Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*Wireless communication, Network topology, Network communications, Distributed networks*

General Terms

Algorithms, Theory

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MIMO, Shannon's theorem, signal-to-noise ratio, channel capacity

1. INTRODUCTION

The broadcast problem is to distribute a single message to all nodes in a wireless network. When the transmission power is limited it seems obvious that the time to deliver the message is lower bounded by the diameter of a graph of wireless point-to-point connections. We show that without additional power we can deliver a message in logarithmic time, if one uses so-called MIMO (multiple input multiple output) communication. In order to establish this result we need an accurate model for radio communication.

1.1 Modeling Radio Communication

The capacity of radio communication is strongly influenced by the locations of the radio stations and this influence has been modeled and studied for a long time in computer science. Since computer scientists concentrate on the algorithmic aspects, the communication models have been chosen mostly for simplicity while neglecting physical accuracy.

In the algorithmic networking community first considerations started with graph based models where nodes represent radio stations and edges indicate connections [3, 11]. There is a long line of research in the area of radio broadcasting which consists of handling the problem of interference in an unknown graph in order to broadcast a message.

Later on, geometric graphs were used to model this communication graph. Starting with unit-disk graphs [4], which reflect the power fading around a transmitter and the signal-to-noise ratio, this model was further developed. For a geometric modeling the delay, energy and dilation of routing in networks have been investigated in [9] and approximative solutions as well as trade-offs between these measures have been presented. Further works [2, 12, 13, 14] also considered variable disk ranges around senders. Different disk sizes indicated safe, unsure and no message receipt. This research was motivated by the path-loss model, where a communication between two radio stations takes place if the signal to interference and noise ratio, defined as $\frac{S}{I+N}$ is larger than a given constant. In this term S denotes the signal power proportional to $d^{-\alpha}$ where d is the distance and α is the path loss exponent, usually in the range between 2 and 5. One can find an overview for different indoor and outdoor path loss models in [19]. The term I denotes the sum of all interfering signal powers approximated as $\sum_{i=1}^{m}(d_i)^{\alpha}$ and N is a constant, which describes the noise coming from the medium and the receiver's hardware. Recent work has described the geometric behavior of senders and interfering radio stations [1] and the complexity of solving communication tasks [6].

Initially, Gupta and Kumar derive in [7] the capacity of a wireless network from the signal-to-noise ratio (SNR). They determine the network throughput for n arbitrarily or randomly placed nodes with single antennas. The path loss exponent is $\alpha > 2$ and they consider interference as boolean property: if the SNR is above a given threshold the signal can be received with a constant not mentioned channel capacity. The capacity is in the order of $\sqrt{n}$. Kumar et al. [8] extend their work from two to three dimensions and then generalize their result for two dimensions by using Shannons law to estimate the channel capacity with $\Theta(\log(1 + \mathrm{SNR}))$.

However this approach completely fails to describe the phenomena with coupled antennas, better known as smart antennas or MIMO (multiple input and multiple output). While there are models that are derived from experimental research in communication, there is little mathematical understanding about these communication models. Furthermore, the relationship of the signal-to-noise ratio in MIMO to the communication bandwidth, which has been theoretically founded for single antennas by Claude Shannon [20], is also not completely understood.

1.2 Related Work

Özgür, Leveque, and Tse show in [17] that linear capacity in the number of nodes n is possible when $\sqrt{A}/\lambda > n$ where A denotes the area A for node placing and λ denotes the wavelength. To achieve linear scaling they use MIMO techniques by allowing several nodes with single antennas to join into a coordinated antenna array. We also arrive at the same precondition $\sqrt{A}/\lambda > m$ for our model to receive a SNR gain of $1/m$ for MIMO and nodes with m antennas in [10]. In [16] they further divide wireless networks in working regimes with the main parameters short-distance SNR, long-distance SNR, and the path loss exponent α of the environment.

In a recent work [15], Özgür et al. present a hierarchical broadcast scheme for n nodes in an one-dimensional network. The basic scheme distributes information in clusters of size M and the beamforming gain of the the M nodes is used to transmit the messages to the target. This recursion step is repeated in a hierarchical strategy. Their analysis assumes a path-loss exponent $1 \leq \alpha < 2$ in the line-of-sight case. At the same time, they demand low SNR $\ll 0$ dB and for small-range communication between neighboring nodes a $\mathrm{SNR}_s \leq n^{\alpha-2}$. In this paper, we assume a path-loss exponent $\alpha = 2$.

On the other hand MIMO is already a standard in use in IEEE WLAN 802.11n and there have been a series of considerations describing the bandwidth gain, which have been theoretically predicted up to a factor of n for n senders and n receivers. However, these predictions rely on the presence of a complex environment where radio signals are reflected from many points which are scattered around. Counterintuitively, such environments are helpful and for sometimes even necessary for MIMO communication. So, we consider here the worst case, which is the free space model.

1.3 Contribution

In this paper, we derive a theoretic model for the communication bandwidth for MIMO. This is based on the fundamentals of physics for electromagnetic waves combined with the theorem of Claude Shannon. We can reduce our model to the standard SINR (signal to interference and noise ratio) model in the plane for the path loss exponent $\alpha = 2$ but with the enhancement of describing bandwidth limits and multiple synchronized senders and receivers. Thereby, we approve the superposition of power of unsynchronized interferences in the SINR model.

Then, we consider the case where multiple senders and receivers are placed on a line. If n senders are synchronized, then the transmission range can be extended by MIMO (multiple input multiple output) communication to a distance of a factor $\sqrt{n}$ compared to a single sender when using the same transmission power of all senders combined. A similar observation can be made for SIMO (single input multiple output) communication. For MIMO we predict a range increase of a factor of n without increasing the power. Regarding the bandwidth only little improvement can be expected if only one pair of MIMO senders and receivers is active. However, it is possible to have multiple communication links in parallel, when multiple MIMO pairs are communicating. From these observations, we propose a broadcasting scheme in $\mathcal{O}(\log n)$ time and the same energy consumption as direct-neighbor communication. We also show the ability of parallelism of this broadcast algorithm in intervals on the line.

2. THE COMMUNICATION MODEL

We consider radio stations R in the plane with antennas oriented perpendicular to a plane and therefore neglect the effects of polarization. The antennas are used for sending and receiving signals, which essentially hold binary strings, called messages. These messages are modulated on the same carrier wave with some frequency f. We assume a line-of-sight communication model without obstacles and neglect the influence of the nodes to the radio communication.

The antennas are placed on a line of this plane. A radio station may use more than one antenna for sending or receiving. The set of antennas of a radio station v are identified by their positions on the line. This position knowledge allows a radio station to send coordinated signals. However, it is also possible that antennas of different radio stations are coordinated.

DEFINITION 2.1. *A set of sending antennas are coordinated if their locations are known at their radio stations, the carrier waves and signal encoding are synchronized and they perform the same task, e.g. send the same message. Receiving antennas are coordinated if they share these properties and the signal can be decoded without further wireless transmission.*

Note that it is conceivable to consider decoding using further wireless communication. However, this causes an increased message size. In the seminal work of [17] and subsequent article [16] this factor is expressed by a so called *observation* of a message. The increase of the message size limits multi-hop routing with uncoordinated MIMO communication because the message size grows exponentially with each hop when observations are forwarded instead of decoding them right away.

We assume that uncoordinated radio stations use unsynchronized clocks, which can be modeled by independently identically distributed random variables describing the relative phase shift of the carrier waves.

The key term to model the reception of a signal is the signal power, while the key of understanding how the signal is transmitted is the electric field. In Appendix A you can find a detailed derivation of our model, which we now shortly summarize.

At the sending antenna u signals are described by the function $s_u(t)$ modulated over a carrier wave described by $a_u\, \mathrm{e}^{j(2\pi f t+\phi_u)}$ where a_u describes the amplitude of the field and ϕ_u some random phase shift (we denote j as the imaginary unit). Now $a_u^2 f^2$ is proportional to the transmission power P and for coordinated senders we can adjust the amplitude and phase shifts to values $a'_u \leq a_u$ and arbitrary ϕ'_u. The complex value $s_u = a'_u\, \mathrm{e}^{j\phi'_u}$ describes the full information of the carrier wave with $|s_u| \leq \sqrt{P}/f$ and transmission power P at antenna u. Note that for coordinated senders these complex numbers can be freely chosen and adjusted.

DEFINITION 2.2. *The electric field E_u at sending antenna u is characterized by $E_u(t) = s_u e^{j2\pi f t}$, where $s_u = |s_u| e^{j\phi}$ describes the amplitude $|s_u| \leq \sqrt{P}/f$ for maximum transmission power P and ϕ the phase shift of the sender u.*

The propagation of the electromagnetic wave is described by the channel matrix which takes two terms into account – the decrease of the electric field of a sinusoidal wave, which is proportional to $\frac{1}{d}$ where d is the distance, and the phase shift induced by the time the signal needs to travel with the speed of light c.

DEFINITION 2.3. *The received electric field $E_v(t)$ at a receiver antenna v is described by*

$$E_v(t) = \sum_{i=1}^{n} s_i \cdot \frac{\mathrm{e}^{j2\pi f(t+|u_i-v|/c)}}{|u_i - v|} = \mathrm{e}^{j2\pi f t} \cdot \sum_{i=1}^{n} s_i \cdot h_{i,k}$$

where $u_1, \ldots, u_n$ are the sender antennas with characteristic scalar s_i.

This modification between sender and receiver can be described by a multiplication with the complex number $h_{i,k} = \frac{\mathrm{e}^{j2\pi f |u_i - v_k|/c}}{|u_i - v_k|}$, where u_i and v_k denote the positions of sender and receiver on the line. The resulting matrix $H = (h_{i,k})_{i,k}$ is called the channel matrix.

Similarly, at the receiver it is possible to amplify and phase shift signals. This is denoted by the multiplication with r_i. However, the amplification also changes the received noise and possibly interfering messages by a factor of $|g_i|$.

So, for (coordinated or uncoordinated) senders $u_1, \ldots, u_n$ the receiver antennas $v_1, \ldots, v_m$ receive an electric field which consists of the linear combination of the signals

$$E(t) = \sum_{i=1}^{n} \sum_{k=1}^{m} g_k \cdot E_{u_i, v_k}(t)$$

where $g_k \in \mathbb{C}$ is chosen for each receiver antenna which consists of a phase shift and an amplification. Furthermore, some noise N will be received which we will take care of soon. Whenever a phase shift occurs the constant factor $2\pi f/c$ occurs. For readability, we omit it and set $2\pi f/c = 1$ for the rest of this paper. We also omit the carrier wave function $\mathrm{e}^{j2\pi f t}$, since the characteristics are solely described by the factors. So, the electric field is no longer a function of time. The power of the coordinated signal can be described by $(E_S)^2$ as well as the power of the uncoordinated signal is $(E_I)^2$. Since we assume independent choices of the phase shifts we can simplify this term.

LEMMA 2.4. *The expected power of uncoordinated senders $w_1, \ldots, w_\ell$ with signal $s'_1, \ldots, s'_\ell$ at the coordinated receivers $v_1, \ldots, v_m$ is*

$$\mathbb{E}[P] = \mathbb{E}\left[|E|^2\right] = \sum_{k=1}^{m} \sum_{i=1}^{\ell} |g_k|^2 \frac{|s'_i|^2}{|w_i - v_k|^2}$$

where g_k denotes the signal gain of the k-th receiver.

The following proof of Lemma 2.4 approves the superposition of power of unsynchronized interferences in the SINR model.

Proof of Lemma 2.4: Given n senders with characteristic scalars $s'_i = a_i\, \mathrm{e}^{j\phi_i}$ and distance $|w_i - r|$ to a receiver r. The sender antennas are not synchronized with phase angle ϕ_i and produce interference at r. The electrical field strength of sender w_i with far field approximation is

$$E_{s_i} = \frac{a_i \cdot \mathrm{e}^{j\phi_i}}{|w_i - r|} \ .$$

The power of the field produced by sender w_i at r alone is

$$P_{w_i,r} = \left|\frac{a_i \cdot \mathrm{e}^{j\phi_i}}{|w_i - r|}\right|^2 = \frac{a_i^2}{|w_i - r|^2} \ .$$

The superposition principle can be applied to the electrical field strength and not to the power.

$$E_r = \textstyle\sum_{i=1}^{n} E_{w_i}$$

The power of the superposed field is then

$$P_r = \left(\textstyle\sum_{i=1}^{n} E_{w_i}\right)^2 \ . \qquad (1)$$

The expected power of the noise is then

$$\begin{aligned}
\mathbb{E}[P] &= \mathbb{E}\left[\left|\sum_{i=1}^{n} \frac{a_i \cdot e^{j\phi_i}}{|w_i - r|}\right|^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \frac{a_i \cdot e^{j\phi_i}}{|w_i - r|} \cdot \sum_{i=1}^{n} \frac{a_i \cdot e^{-j\phi_i}}{|w_i - r|}\right] \\
&= \mathbb{E}\left[\sum_{i \in \{1..n\}} \left(\frac{a_i}{|w_i - r|}\right)^2\right] \\
&\quad + \underbrace{\mathbb{E}\left[\sum_{i=1}^{n} \sum_{k=1, i \neq k}^{n} e^{j(\phi_i - \phi_k)}\right]}_{=0} \frac{a_i}{|w_i - r|} \cdot \frac{a_k}{|w_k - r|} \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \frac{a_i^2}{|w_i - r|^2}\right] .
\end{aligned}$$

□

This corresponds to the well known SINR model. The noise power is amplified like all other signals at the m coordinated receivers by a factor of $|g_k|^2$ with signal gain g_k at the k-th receiver. Since electric fields superpose we get the following signal-to-noise ratio.

DEFINITION 2.5. *For n coordinated senders at positions $u_1, \ldots, u_n$ and m coordinated receivers at positions $v_1, \ldots, v_m$ the signal-to-noise-ratio (SINR) can be determined as*

$$SINR = \frac{\left|\sum_{i=1}^{n} \sum_{k=1}^{m} s_i \cdot \frac{e^{j|u_i - v_k|}}{|u_i - v_k|} \cdot g_k\right|^2}{\sum_{k=1}^{m} |g_k|^2 \left(N + \sum_{i=1}^{\ell} \frac{P'_i}{|w_i - v_k|^2}\right)} = \frac{|s \cdot H \cdot g|^2}{N' + I}$$

where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is the set of the coordinated sending antenna positions, $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ is the set of the coordinated receiving antenna positions, and $w = (w_1, \ldots, w_\ell) \in \mathbb{R}^\ell$ is the set of uncoordinated sender antenna positions on the line. N is the power equivalent of the noise level at each receiving antenna v_k. P'_i describes the power of the interfering antenna w_i. The positions of the coordinated senders and receivers describe the channel matrix in free space

$$H_{i,k} = \frac{e^{j|u_i - v_k|}}{|u_i - v_k|} \quad i \in [n],\, k \in [m] .$$

$$I = \sum_{i \in [\ell], k \in [m]} |g_k|^2 \frac{P'_i}{|w_i - v_k|^2}$$

is the sum of the received signal power from uncoordinated senders. And the received noise is given by

$$N' = N \sum_{k \in [m]} |g_k|^2 .$$

The vectors $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ with $|s_i|^2 \leq P_{u_i}$ and $g = (g_1, \ldots, g_m) \in \mathbb{C}^m$ can be chosen arbitrarily.

For the bandwidth we follow Shannon's theorem. We also assume that a minimum SINR is necessary to establish communication.

DEFINITION 2.6. *If the SINR is above a certain threshold, then communication can be established. For higher SINR values the bandwidth of the transmission is modeled by $f \cdot \log(1 + SINR)$ where f is the carrier frequency.*

The *threshold effect* [20] causes the error rate to increase drastically when the noise is over a certain threshold of the system design.

3. POWER GAIN AND DELIBERATE ATTENUATION OF COORDINATED SENDERS IN MISO

For senders on a line, this results in a power gain when the senders are in sync, e.g. all senders emit the same signal and a receiver on the line receives the signal of all senders in the same phase (Theorem 3.1). The coordinated senders produce noise in the opposite of the receiver's direction and we can also attenuate this noise by increasing the path loss exponent at the expense of decreasing power gain towards the receiver. We call this *deliberate attenuation* (Theorem 3.3).

THEOREM 3.1. *If n coordinated senders send with transmission power $P_i = P/n$ each, at positions $u_i \geq 0$, they can always produce a signal at a receiver $v_k \geq \max_\ell\{u_\ell\}$, which is n times more powerful than a single sender at position 0 with transmission power P when m receiver antennas are positioned at $v_1, \ldots, v_m \geq u_n$.*

PROOF. Choose $s_i = e^{ju_i}\sqrt{P/n}$ and $g_k = e^{-jv_k}$. This results in

$$\begin{aligned}
\mathrm{SINR}_{n,m} &= \frac{\left|\sum_{i=1}^{n} \sum_{k=1}^{m} e^{ju_i}\sqrt{P/n} \cdot \frac{e^{j|u_i - v_k|}}{|u_i - v_k|} e^{-jv_k}\right|^2}{\sum_{k=1}^{m} |e^{-jv_k}|^2 \cdot (N + I)} \\
&= \frac{\frac{P}{n} \cdot \left|\sum_{i=1}^{n} \sum_{k=1}^{m} \frac{1}{|u_i - v_k|}\right|^2}{m \cdot (N + I)} \\
&\geq \frac{n \cdot P \cdot \left|\sum_{k=1}^{m} \frac{1}{|v_k|}\right|^2}{m \cdot (N + I)} .
\end{aligned}$$

The same equation for one sender at the origin yields for $s_1 = \sqrt{P}$ and $g_k = e^{-jv_k}$.

$$\begin{aligned}
\mathrm{SINR}_{1,m} &= \frac{\left|\sum_{k=1}^{m} \sqrt{P} \cdot \frac{e^{j|v_k|}}{|v_k|} \cdot e^{-jv_k}\right|^2}{\sum_{k=1}^{m} |e^{-jv_k}|^2 (N + I)} \\
&= P \frac{\left|\sum_{k=1}^{m} \frac{1}{v_k}\right|^2}{m(N + I)} \\
&\leq \frac{1}{n} \mathrm{SINR}_{n,m} .
\end{aligned}$$

□

This implies that although the overall transmission power is the same, the signal range with n coordinated antennas extends by a factor of $\sqrt{n}$. This phenomenon is long known and is called power gain in MISO [21].

COROLLARY 3.2. *Any n coordinated senders can send $\sqrt{n}$ times farther than a single sender consuming the same power.*

This is not contradicting the principle of conservation of power, since we consider only the power on the line, whereas the power distribution in the rest of the space changes drastically.

While the factor n power gain is well known, the observation, that one can deliberately attenuate the signal in one direction, is new to our knowledge.

THEOREM 3.3. *Any n coordinated antennas in general positions on the line can produce a fast fading signal on the line which decreases with SINR $\mathcal{O}\left(1/d^{2n}\right)$ in distance d.*

PROOF. To increase the path-loss exponent to $\alpha = 2n$ or a field strength decreasing with $1/d^n$ we want to ensure that

$$h\left(x\right) = \sum_{i=1}^{n} s_i \cdot \frac{\mathrm{e}^{j(x-x_i)}}{x - x_i} = \frac{\gamma}{\mathcal{O}\left(x^n\right)}$$

for the complex antenna characteristics s_i and some constant γ. W.l.g. we only consider x-values outside the sender group with $x > x_i$. We extend all summands to the same denominator and the goal is to simplify the nominator to a constant γ to decrease the signal strength to $\mathcal{O}\left(x^{-n}\right)$.

$$\begin{aligned} h\left(x\right) &= \sum_{i=1}^{n} \frac{s_i\,\mathrm{e}^{j(x-x_i)} \cdot \prod_{k=1,k\neq i}^{n}\left(x - x_k\right)}{\left(x - x_i\right) \cdot \prod_{k=1,k\neq i}^{n}\left(x - x_k\right)} \\ &= \frac{\sum_{i=1}^{n} s_i\left(d_{0,i}x^{n-1} + d_{1,i}x^{n-2} + \cdots + d_{n-1,i}\right)}{\prod_{k=1}^{n}\left(x - x_k\right)} \quad (2) \\ &= \frac{\gamma}{\prod_{k=1}^{n}\left(x - x_k\right)} \quad (3) \end{aligned}$$

There is a choice for $(s_1, \ldots, s_n)$ resolving (2) to (3), since there is a solution to the following equation

$$\begin{pmatrix} d_{0,1} & \ldots & d_{0,n} \\ \cdots & & \cdots \\ d_{n-1,1} & \ldots & d_{n-1,n} \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \end{pmatrix} .$$

Because n vectors of length $(n-1)$ are linear dependent there is always a non-trivial solution $(s_1, \ldots, s_n) \neq (0, \ldots, 0)$ to this equation.

□

One may object that the neglected near-field components have stronger asymptotics than this attenuated signal. However, this proof technique also applies to a more accurate model chosen, which reflects far-field and near-field, and yields the same result, i.e. a near-field component for the electromagnetic field of $\mathcal{O}\left(\frac{1}{d^2}\right)$ with distance d to the sender (see Equation 5 in the appendix).

THEOREM 3.4. *Given $n = \rho \cdot \beta$ (for $\rho \geq 2$) coordinated senders with power P each, we can obtain in opposite directions a power gain $\mathcal{O}\left(\beta P/d^2\right)$ and a deliberate power attenuation of $\mathcal{O}\left(\beta P/d^{2\rho-2}\right)$.*

Proof Sketch. Let $u = (u_1, \ldots, u_n)$ be the sender antennas divided into β groups of ρ antennas.

Each group of the ρ antennas will deliberately attenuate for a position $x > \max\{u_i\}$ and distance $d = (x - \max\{u_i\})$ with a power of $\mathcal{O}(\frac{P}{d^{2\rho-2}})$ and send a signal of power of at least P/d^2 to a position $x < \min\{u_i\}$ and the distance $d = (\min\{u_i\} - x)$ to the group of antennas. In order to achieve this claim the linear equation in the proof of Theorem 3.3 needs to be combined with an inequality preventing the attenuation to the left. The probabilistic method shows that the additional degree of freedom by attenuating only to $O(P/d^{2\rho-2})$ allows this property. Now we multiply each of the β groups with a random phase shift.

By the argument of Lemma 2.4 the expected power will be the sum of all signal powers of the sub-groups. From this, we can induce the existence of a choice such that the signal is attenuated by $O(\rho\beta/d^2)$ to the left and $\mathcal{O}\left(\beta P/d^{2\rho-2}\right)$ to the right. □

COROLLARY 3.5. *Among many others the following combination of deliberate attenuation and power gain to different directions are possible for antennas with power P each.*

1. *$\mathcal{O}\left(P/d^2\right)$ to the left and $\mathcal{O}\left(P/d^{2n-2}\right)$ to the right ($\rho = n$, $\beta = 1$)*
2. *$\mathcal{O}\left(\sqrt{n}P/d^2\right)$ to the left and $\mathcal{O}\left(\sqrt{n}P/d^{2\sqrt{n}-2}\right)$ to the right ($\rho = \sqrt{n}$, $\beta = \sqrt{n}$)*
3. *$\mathcal{O}\left(nP/d^2\right)$ to the left and $\mathcal{O}\left(nP/d^{2c}\right)$ to the right for any integer $c \geq 1$ ($\rho = c + 1$, $\beta = n/(c + 1)$).*

4. POWER GAIN AND DIVERSITY GAIN OF SIMO AND MIMO

For a single sender we can also experience a power gain of a factor n if we use coordinated antennas for receiving (SIMO).

THEOREM 4.1. *Given m coordinated receivers $0 < v_1 < \ldots < v_m$ and a sender at the origin 0. Let $SINR_{1,m}$ be the SINR of these receivers and let $SINR_{1,1}$ be the SINR of a single receiver at v_m. Then, $SINR_{1,1} \leq \frac{1}{m} SINR_{1,m}$.*

PROOF. Choose $s_1 = \sqrt{P}$ and $g_k = \mathrm{e}^{-j2\pi f v_k}$. This results in

$$\begin{aligned} \mathrm{SINR}_{1,m} &= \frac{\left|\sum_{k=1}^{m} \sqrt{P} \cdot \frac{\mathrm{e}^{j|v_k|}}{|v_k|} \cdot \mathrm{e}^{-jv_k}\right|^2}{\sum_{k=1}^{m} \left|\mathrm{e}^{-jv_k}\right|^2 (N + I)} \\ &= P \cdot \frac{\left|\sum_{k=1}^{m} \frac{1}{|v_k|}\right|^2}{m\left(N + I\right)} \\ &\geq m \frac{P}{(v_m)^2 \cdot (N + I)} \\ &= m \cdot \mathrm{SINR}_{1,1} \end{aligned}$$

□

Again this results in an extension of the transmission range.

COROLLARY 4.2. *Any n coordinated receivers can get a message from a distance $\sqrt{n}$ times farther than a single receiver.*

With the same calculation one can see a power gain in MIMO.

THEOREM 4.3. *Given n coordinated senders $u_1 < \ldots < u_n < 0$ and m coordinated receivers $0 < v_1 < \ldots < v_m$. Let $SINR_{n,m}$ be the SINR of these senders and receivers and let $SINR_{1,1}$ be the SINR of a single sender at u_1 and a single receiver at v_m. Then, $SINR_{n,m} \geq nm \cdot SINR_{1,1}$.*

These power gains help to extend the communication reach. However, there is also a direct possibility to increase the bandwidth using the so-called diversity gain.

It is often mentioned in literature (e.g. in [18]) that angular spread is essential for MIMO transmission. Our first observation is that in principle such a diversity gain is possible on the line even in free space.

Lemma 4.4. *For coordinated senders $u_1 < \ldots < u_n$ and coordinated receivers $v_1 < \ldots < v_m$ on a line with $u_n < v_1$ or $v_m < u_1$ the channel matrix H has rank $\min\{n, m\}$.*

Proof. Without loss of generality we consider the only the case $u_n < v_1$. Let $n = m$, then the channel matrix is

$$\begin{aligned} H &= \left(\frac{\mathrm{e}^{j|u_i - v_k|}}{|u_i - v_k|}\right)_{i,k\in[n]} = \left(\frac{\mathrm{e}^{j(v_k - u_i)}}{v_k - u_i}\right)_{i,k\in[n]} \\ &= D\left(\left(\mathrm{e}^{-ju_i}\right)_{i\in[n]}\right)\left(\frac{1}{v_k - u_i}\right)_{i,k\in[n]} D\left(\left(\mathrm{e}^{jv_k}\right)_{k\in[n]}\right) \end{aligned}$$

where $D(a)$ denotes the diagonal matrix of vector a, which has full rank if a has no zero entry. The matrix $\left(\frac{1}{v_k - u_i}\right)_{i,k\in[n]}$ is a Cauchy matrix and thus is invertible for all u, v if for all i, k: $u_i \neq v_k$. $\square$

Theorem 4.5. *For coordinated senders $u_1 < \ldots < u_n$ and non coordinated receivers $v_1 < \ldots < v_m$ with $m < n$ on a line with $u_n < v_1$ or $v_m < u_1$ it is possible to send to any subset of receivers without producing a signal at the other receivers.*

Proof. Consider the vector $a_1, \ldots, a_m$ such that $a_i = 1$ if i is in the subset of aimed receivers and $a_i = 0$ otherwise. Now, we use only m senders. Then let H^{-1} be the inverse of H, which exists because of Lemma 4.4. Then each sender u_i uses the parameter $qH^{-1}a$, where $q = P/\max\{|(H^{-1}a)_i|\}$, where P denotes the maximum possible transmission power. The resulting signal is therefore $qHH^{-1}a = qa$. $\square$

Using this theorem it is possible to send n messages in parallel from n coordinated senders to n uncoordinated receivers, which can be seen as parallel MISO. For this, we choose a receiver and modulate a signal, which can be received at this receiver only, while the other receivers get no signal. Now, we repeat this for all receivers and send the superposed signal from the n coordinated senders. As a result each uncoordinated receiver gets only "his message". This seems to increase the bandwidth between senders and receivers on the line by a factor of n. However, the delimiting factor is the attenuation of the signals imposed by the maximum transmission power P and the entries of the inverse channel matrix H^{-1}.

Lemma 4.6. *Fix a set of n senders $u_1, \ldots, u_n$ and n receivers $v_1, \ldots, v_n$. Consider the channel matrix of u and $(v_1 + d, \ldots, v_n + d)$ for increasing distance d on the line. Then, the maximum absolute value of the inverse of the channel matrix is $\Theta(d^{2n-1})$.*

Proof. The absolute values of the channel matrix are described by the Cauchy matrix

$$M = \left(\frac{1}{v_k - u_i}\right)_{i,k\in[n]} .$$

The determinant of a Cauchy matrix is

$$\begin{aligned} \det M &= \frac{\prod_{i=2}^{n}\prod_{k=1}^{i-1}(v_i - v_k)(u_i - u_k)}{\prod_{i=1}^{n}\prod_{k=1}^{n}(v_i - u_k)} \\ &= \frac{\prod_{i=2}^{n}\prod_{k=1}^{i-1}(v_i - v_k)(u_i - u_k)}{\prod_{i=1}^{n}\prod_{k=1}^{n}(d + v_i - u_k)} \\ &= \Theta\left(\frac{1}{d^{n^2}}\right) . \end{aligned}$$

The inverse $D = (d_{ik})_{i,k\in[n]}$ of a matrix can be computed as

$$d_{ik} = (-1)^{i+k}\frac{\det(M_{ik})}{\det(M)}$$

where M_{ik} is the submatrix of M without the i-th row and k-th column. Note that M_{ik} is also a Cauchy matrix. Therefore

$$|d_{ik}| = \Theta\left(\frac{d^{n^2}}{d^{(n-1)^2}}\right) = \Theta\left(d^{2n-1}\right)$$

$\square$

So, the usage of Theorem 4.5 leads to an attenuation by a factor of $O(1/d^{2n-2})$, which is close to the deliberate attenuation which we have discussed before. On the positive side, we show that it is possible to send n message in parallel from n coordinated senders to n uncoordinated receivers even in free space. However, the power of each antenna must be chosen extremely large with respect to the noise, interference power, and distance, i.e. $P \geq (N + I)d^{4n+2}$. For such powerful senders the diversity gain of MIMO is larger than the bandwidth increase using the classic Shannon bounds even in the free space communication model.

5. BROADCASTING ON A LINE

We now concentrate on the main problem, which is the broadcast problem where n nodes with one antenna each are placed equidistantly on a line. Assume the first node of the line is the originator of the broadcast message. The broadcast scheme works in rounds. In the first round the only informed node u_1 transmits the message to neighbor u_2. The informed node synchronizes with the first node and thus becomes coordinated. In the subsequent rounds all coordinated senders use the MISO power gain to reach the next neighbors and synchronize them. This process continues until all nodes are informed. Using our previous observations of the MISO power gain we can prove that this way the number of coordinated nodes increases exponentially inducing a logarithmic time for the broadcast.

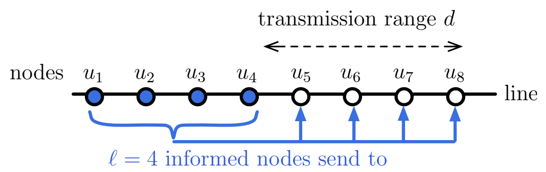

Figure 1: Four coordinated nodes $u_1, .., u_4$ broadcast and double the the number of informed nodes.

Theorem 5.1. *The broadcast problem of n equidistant nodes on a line where each node can establish a point-to-point connection to each neighbor, can be solved in time $\mathcal{O}(\log n)$ and energy $\mathcal{O}(n)$ using MISO and wireless self-coordination.*

Proof. Without loss of generality the nodes have unit distance. For a given noise N and a required threshold SNR_0, the minimum power P_0 to reach a neighbor in unit distance is $P_0 \geq N \cdot \mathrm{SNR}_0$.

First let us analyze the transmission range d of ℓ adjacent, coordinated, and informed nodes. Each informed node u_i uses the characteristic $s_i = \sqrt{P_0}e^{-ji}$. If all ℓ nodes send with unit power P_0 the signal power they produce in distance d is

$$
\begin{aligned}
|h(\ell, d)|^2 &= \left| \left(\sum_{i=1}^{\ell} \underbrace{\sqrt{P_0} \cdot \mathrm{e}^{-ji}}_{s_i} \cdot \frac{\mathrm{e}^{j(d+i)}}{d+i} \right)^2 \right| \\
&= P_0 \cdot \left(\sum_{i=1}^{\ell} \frac{1}{d+i} \right)^2 \\
&= P_0 \cdot (\Psi(d+\ell+1) - \Psi(d+1))^2 \\
&> P_0 \cdot \ln((d+\ell)/d)^2 \ .
\end{aligned}
$$

where $\Psi(x)$ is the digamma function.

Now, if $\frac{|h(\ell,d)|^2}{N} \geq \mathrm{SNR}_0$, then the receiver in distance d gets the message and can be coordinated.

$$
\frac{|h(\ell, d)|^2}{N} \geq \mathrm{SNR}_0 \cdot \ln((d+\ell)/d)^2
$$

So, for $\ln((d+\ell)/d)^2 \geq 1$ the node in distance d can be reached. This is the case for $d \leq \frac{1}{e-1}\ell$. If the number of informed and coordinated nodes in round i is ℓ_i then in the next round

$$
\ell_{i+1} \geq \ell_i + \max\left\{1, \left\lfloor \frac{1}{e-1}\ell_i \right\rfloor \right\}
$$

nodes are informed. Clearly $\ell_i = \Omega(\kappa^i)$ for any $\kappa < \frac{\mathrm{e}}{\mathrm{e}-1}$. Hence, after $T = \mathcal{O}(\log n)$ rounds all n nodes are informed.

The energy is bounded by

$$
\begin{aligned}
\mathcal{O}\left(\sum_{i=1}^{T} P_0 \ell_i \right) &= \mathcal{O}\left(P_0 \sum_{i=1}^{T} \kappa^i \right) \\
&= \mathcal{O}\left(P_0 \sum_{i=0}^{T-1} \frac{n}{\kappa^i} \right) \\
&= \mathcal{O}(n P_0) \ .
\end{aligned}
$$

□

An interesting feature of this broadcasting process is that it can be performed in parallel, since we can bound the interfering energy by the following theorem.

THEOREM 5.2. *For an infinite number of equidistant nodes on the line the broadcasting algorithm above can be performed for each contiguous group of n nodes if the minimum distance between these groups is $\mathcal{O}(n)$.*

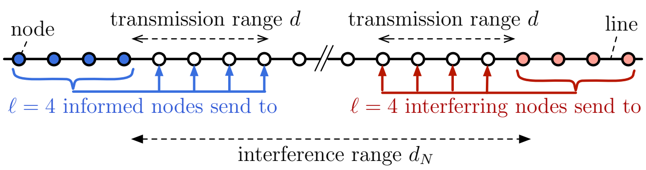

Figure 2: parallel transmissions: $\ell = 4$ nodes (blue) send to range d with interference in distance d_N (red).

The following proof shows Theorem 5.2 that the noise produced by unsynchronized simultaneous sending antenna groups is independent from the number of nodes in the network n.

Proof of Theorem 5.2: Let $\ell \leq n$ be the number of active senders and let d denote the distance between the groups. Using the upper bound for the signal strength of one sender group with ℓ antennas with $(s \cdot \ell / d)$ we get a noise level of

$$
|h_N| \leq \sum_{i=0}^{\infty} \frac{s \cdot \ell}{i \cdot d} \mathrm{e}^{j\beta_i} = \frac{\ell}{d_N} \cdot \sum_{i=0}^{\infty} \frac{\mathrm{e}^{j\beta_i}}{i} = \frac{\ell}{d_N} \cdot c_N \ .
$$

Let $\overline{c_N}$ denote the complex conjugate of c_N.

$$
\begin{aligned}
|c_N|^2 &= c_N \cdot \overline{c_N} \\
&= \sum_{i=1}^{\infty} \frac{\mathrm{e}^{j\beta_i}}{i} \cdot \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-j\beta_k}}{k} \\
&= \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right) + \left(\sum_{i=1}^{\infty} \sum_{k=1, i \neq k}^{\infty} \frac{\mathrm{e}^{j(\beta_i - \beta_k)}}{i \cdot k} \right) \\
&= \frac{\pi^2}{6} + \sum_{i=1}^{\infty} \sum_{k=1, i \neq k}^{\infty} \frac{\mathrm{e}^{j(\beta_i - \beta_k)}}{i \cdot k}
\end{aligned}
$$

For each index tuple (i, k) with $i \neq k$ there exists a symmetric (k, i) with the negated imaginary value.

$$
\forall i \neq k : \quad \Im\left(\mathrm{e}^{j(\beta_i - \beta_k)} \right) + \Im\left(\mathrm{e}^{j(\beta_k - \beta_i)} \right) = 0
$$

So, we get only a sum of real numbers.

$$
\sum_{i=1}^{\infty} \sum_{k=1, i \neq k}^{\infty} \frac{\mathrm{e}^{j(\beta_i - \beta_k)}}{i \cdot k} = \sum_{i=1}^{\infty} \sum_{k=1, i \neq k}^{n} \frac{\cos(\beta_i - \beta_k)}{i \cdot k}
$$

We have assumed that angles $\beta_i \in [0, 2\pi)$ are independently, identically, and uniformly distributed over $[0, 2\pi)$. So the expectation of $\cos(\beta_i)$ is $\left(\frac{1}{2\pi} \int_{\beta=0}^{2\pi} \cos\beta \, \mathrm{d}\beta \right) = 0$ And, the expected value of the sum is

$$
\mathbb{E}\left[|c_N|^2\right] = \frac{\pi^2}{6} + \sum_{i=1}^{\infty} \sum_{k=1, i \neq k}^{\infty} \frac{\overbrace{\mathbb{E}[\cos\beta_i - \beta_k]}^{0}}{i \cdot k} = \frac{\pi^2}{6} \ .
$$

The root mean square of h_N is therefore

$$
|h_N|_{\mathrm{rms}} = \frac{\ell}{d_N} \frac{\pi}{\sqrt{6}} = \mathcal{O}\left(\frac{\ell}{d_N} \right) \ . \qquad (4)
$$

□

Figure 3 illustrates the result of Lemma 5.2 for the noise strength $|h_N|_{\mathrm{rms}} = \frac{\ell}{d_N} \frac{\pi}{\sqrt{6}}$. In the experiment, $\frac{\ell}{d_N}$ was set to 1 and the phase angels of the interfering sender groups are chosen uniform at random with $\beta_i \in [0, 2\pi)$. Each number of

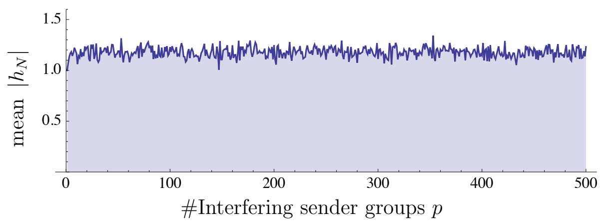

Figure 3: Experimental result for the signal strength $|h_N|$ varying the number of sender groups p.

interfering sender groups was tested 100 times and averaged.

The total average measured strength of noise was ≈ 1.17 whereas the factor in the proof is $\frac{\pi}{\sqrt{6}} \approx 1.28$.

6. CONCLUSION

We present a communication model for the bandwidth in MIMO communication in free space which enhances the SINR model. Exploiting MISO power gain of n coordinated senders increases the transmission power by factor n compared to a single sender with same power. We show how to obtain a deliberate attenuation in MISO for n coordinated senders on a line with factor $1/\mathcal{O}\left(d^{2n}\right)$. While theoretically diversity gain of MISO/MIMO communication with n independent data streams is possible, we show that this works in the free space model only for high SINR and for short distances. We present a logarithmic time MISO broadcast scheme for n nodes placed on a line in time $\mathcal{O}(\log n)$ which needs only a constant times more energy than the sequential direct-neighbor communication. We show that this algorithm broadcasting to $\mathcal{O}(n)$ nodes does not influence simultaneous broadcasts which are in distance $\Omega(n)$.

Note that the communication model is deduced from analyzing Hertz dipole antennas and basic physical observation. So, the model has a solid foundation, which can be understood even by non-physicists. The complete derivation can be found in the Appendix.

Outlook

A straight-forward question is how this approach generalizes to two dimensions. This question is not easy to be answered. The power gain in MISO and SIMO results from the beam forming. In a preceding paper we have analyzed the angle of the main beam [10] for the case of randomly placed antennas in a disk of radius r. It turns out that besides the main beam a small number of side beams appears next to it, while in the residual directions the power behaves like a Gaussian distribution. A next step is an understanding of free space MIMO effects in equidistant grid networks, and how broadcasting can be improved.

Note that the polarization does not play a role in the two-dimensional case, since we assume antennas perpendicular to the plane. Clearly this effect cannot be ignored anymore in three dimensions. Then, the channel matrix cannot be described by a single complex value. A more sophisticated model is needed, which is also part of future research.

In this work we have only considered the free space model. It is well known that the channel matrix allows a diversity gain, if the environment provides reflections and multiple path diffraction. The relationship of the environment to the channel matrix is not fully understood so far. So, many research papers simply assume the best possible channel matrix, the existence of which is not clear. Here, further research may help to a better understanding of the influence of a limited number of obstacles to the possible positive impact on communication.

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APPENDIX

A. APPENDIX: DERIVATION OF THE COMMUNICATION MODEL

In this appendix, we present the physical basis for our communication model of Section 2 starting at electromagnetic fields of antennas to data transmission, and the derived transmission capacity.

A.1 Electric Fields

We briefly summarize essentials for radio communication based on Maxwell's equations. You can find the following observations in much greater detail in Physics textbooks. Here, we now present a compilation of "The Feynman Lectures on Physics" Vol. I. [5], chapter 28 and 29. An electric field **E** is a vector at each point describing the force on a charged particle. It is described for a single particle with charge q as

$$\mathbf{E} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{e_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{\mathbf{r}'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{\mathbf{r}'}\right) \tag{5}$$

where $c \approx 3.000 \times 10^6 m/s$ is the speed of light, and $\varepsilon_0 \approx 8.854 \times 10^{-12} F/m$ is the electric constant, r' is the distances to the particle where it has been considering the speed of light and the distance. $\mathbf{e}_{\mathbf{r}'}$ denotes the unit factor in the direction.

Note that this equation already combines the electric and magnetic field which is described by (the Maxwell-Faraday equation)

$$\mathbf{B} = -\mathbf{e}_{\mathbf{r}'} \times \mathbf{E}/c\ .$$

In the far-field for large distances r the last component in Eq. 5 prevails, rendering the equation to

$$\mathbf{E} = \frac{-q}{4\pi\varepsilon_0 \cdot c^2} \cdot \frac{d^2}{dt^2}\mathbf{e}_{\mathbf{r}'}\ . \tag{6}$$

If a particle moves vertically along a line according where charged particles are moved with acceleration function $a(t)$ the electric field has an approximated magnitude of

$$E(t) = \frac{-q}{4\pi\varepsilon_0 \cdot c^2 r} \cdot \sin\theta \cdot a(t - r/c)$$

and the orientation of the vector is as been shown in Figure 4.

The electric fields have the superposition property. For two electric fields $\mathbf{E_1}$ and $\mathbf{E_2}$ the resulting electric field is

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2}\ . \tag{7}$$

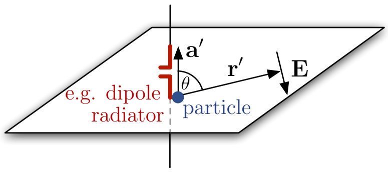

Figure 4: Electric field E in distance r′ of a dipole radiator in the plane with a charge q accelerated with a.

This of course applies for single charges in antennas as well as the different currents of multiple antennas.

Radio signals are modulated as sine curves with $x_0 \cos\omega t$, which leads to an acceleration of

$$a(t) = -\omega^2 x_0 \cos\omega t = a_0 \cos\omega t$$

where $\omega = 2\pi f$ with frequency f, x_0 is the amplitude of the charged particle, ω denotes the frequency, and $a_0 = -\omega^2 x_0$. This results in the well-known Hertz dipole equation

$$E(t) = \frac{-q}{4\pi\varepsilon_0 \cdot c^2 r} \cdot \sin\theta \cdot a_0 \cos\omega(t - r/c)\ . \tag{8}$$

Of course the orientation of the antennas plays a major role. However, if we restrict ourselves to a two-dimensional plane with perpendicular antennas all electric fields are oriented perpendicular to the plane. This allows us to simplify the dynamic electric field to a scalar field. The far-field approximation of Equation 6 holds for a distance r of a few wavelengths $\lambda = c/f$.

A.2 Power

The power or energy per second of an electric wave through an unit area is

$$S = \varepsilon_0 c \cdot E^2$$

with the impedance of free space $1/(\varepsilon_0 c) \approx 376.7$ ohms. Thus, the power increases inversely to the square of the distance with

$$S = \frac{q^2 \cdot a(t)^2 \cdot \sin^2\theta}{16\pi^2\varepsilon_0 \cdot r^2 \cdot c^3}\ .$$

Therefore the power through the enveloping surface of radius r produced of a charge q oscillating with ω is

$$P = \frac{q^2\omega^4 x_0^2}{12\pi\varepsilon_0 c^3}\ .$$

The length of the antenna is described by x_0 which is proportional to $c/f = 2\pi c/\omega$.

At a receiver antenna parallel to the movement of the particle this causes a voltage proportional in E. Also the current is proportional according to Ohm's law, however the inductances plays a major role. Summarizing we observe that the received power P at the antenna is

$$P = kE^2 \tag{9}$$

where k is a suitable constant for a fixed frequency. This also holds for the combination of antennas, since the electric fields increase each voltage and each current.

This leads to two interesting observations, which has been proved useful in antenna design for a long time.

1. Two antennas in sync produce an electric field twice the size. So, four times the power arrives at the receiver antenna.
2. Two receivers can reproduce four times the power of a sender antenna if the induced current is time shifted accordingly.

A.3 Modulation

This observation is only possible if one carefully considers the interplay of the locations of the antennas and the time shift to achieve the constructive interference. For this we introduce the following notations. Let $\mathbf{s_1}, \ldots, \mathbf{s_n}$ denote the locations of sender antennas in two dimensional space, likewise $\mathbf{r_1}, \ldots, \mathbf{r_n}$ denote the receiver antennas.

If we assume a amplitude/phase shift key modulation with function $a_{\mathrm{am}}(t)$, $a_{\mathrm{pm}}(t)$, so the movement of a particle can be described as:

$$a_{\mathrm{am}}(t) \cdot \cos(\omega t + a_{\mathrm{pm}}(t)) = \Re\left(a_{\mathrm{am}}(t) \cdot \mathrm{e}^{j a_{\mathrm{pm}}(t)} \mathrm{e}^{j\omega t}\right)$$

For simplicity we use the complex number

$$a(t) = a_{\mathrm{am}}(t) \cdot e^{j a_{\mathrm{pm}}(t)}$$

as the combined signal emitted at s over time (while $\mathbf{s}$ denotes the location).

A.4 Superposition

So, at receiver r we have the following electric field for one sender s.

$$\begin{aligned} E &\propto \Re\left(a(t) \cdot \frac{\mathrm{e}^{j\omega(t-|\mathbf{s}-\mathbf{r}|/c)}}{|\mathbf{s}-\mathbf{r}|}\right) \\ &= \Re\left(a(t) \cdot \frac{\mathrm{e}^{-j\omega|\mathbf{s}-\mathbf{r}|/c}}{|\mathbf{s}-\mathbf{r}|} \cdot \mathrm{e}^{j\omega t}\right) \end{aligned}$$

We denote by $h_{i,k} = \mathrm{e}^{-j\left|\mathbf{s_i}-\mathbf{r_j}\right|/c} \cdot |\mathbf{s_i} - \mathbf{r_k}|^{-1}$ the amplitude shift transformation of the signal from sender s_i to receiver r_k. Hence, the electric field received at receiver r_k is therefore

$$E_k \propto \Re\left(\sum_{i=1}^{n} a_i(t) \cdot h_{i,k} \cdot \mathrm{e}^{j\omega t}\right) .$$

This describes the MISO case (multiple input/single output). For true MIMO all antennas are combined. This could be a simple addition of the electric fields. However, more likely is that a time shift t_k and a dampening $d_k \leq 1$ is applied. These terms can be adjusted to increase the sensitivity and we combine these terms to $r_k = d_k e^{j t_k}$.

Now the combined received electrical field is

$$E = \sum_{k=1}^{m} E_k = \Re\left(\sum_{k=1}^{m}\sum_{i=1}^{n} a_i(t) \cdot h_{i,k} \cdot r_k \cdot e^{j\omega t}\right) . \quad (10)$$

A.5 Signal to Interference + Noise Ratio (SINR)

For the successful radio reception of the information in $a(t)$ from sender s_i to receiver r, the magnitude of the also undesirably received noise is crucial. The standard measure in literature is the Signal-to-Noise Ratio (SNR) or Signal-to-Interference+Noise Ratio (SINR). The second measure SINR (see [7]) also includes – besides environmental noise and noise in the receiver (e.g. amplifier stage) N – the noise produced by interfering senders s_k involved in parallel transmissions

$$\mathrm{SINR}(r) = \frac{\frac{P(s_i)}{|\mathbf{s_i}-\mathbf{r}|^\alpha}}{N + \sum_{k\neq i} \frac{P(s_k)}{|\mathbf{s_k}-\mathbf{r}|^\alpha}} . \quad (11)$$

The path-loss exponent is here α which is $\alpha = 2$ for free-space and $\alpha > 2$ including obstacles absorbing the energy. The power P is in Eq. 11 exclusive the path-loss factor. Here we answer the question why we sum up the receive power when the superposition principle is applied for the field strengths.

A.6 Data Rate in Presence of Noise

The maximum possible data rate in a channel with white noise was derived by Shannon in 1949 [20]. In the presence of noise with power N, each modulation scheme with power P uses a limited number of distinguishable signals. The maximum power of the received signal is then $(P+N)$ and since each transmitted signal can be perturbed by noise power N there are $K \cdot \sqrt{(P+N)/N} = K \cdot \sqrt{1+P/N}$ distinguishable signals for some constant K near unity. Given a bandwidth W in Hertz we can transmit in a time unit up to $\left(K\sqrt{1+P/N}\right)^{2W}$ distinct signals. Since we can decode in m possible states $\log_2 m$ binary digits this gives a maximum capacity in bits per second of the well-known Shannon-Hartley theorem

$$C = W \cdot \log_2\left(1 + \frac{P}{N}\right) = W \cdot \log_2(1 + \mathrm{SINR}) . \quad (12)$$

For a low SNR ≈ 0 we can approximate

$$C \approx W \cdot \mathrm{SINR} \cdot \log_2(\mathrm{e}) . \quad (13)$$