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Algorithms for Radio Networks

Ideal Cells: Voronoi-Diagrams

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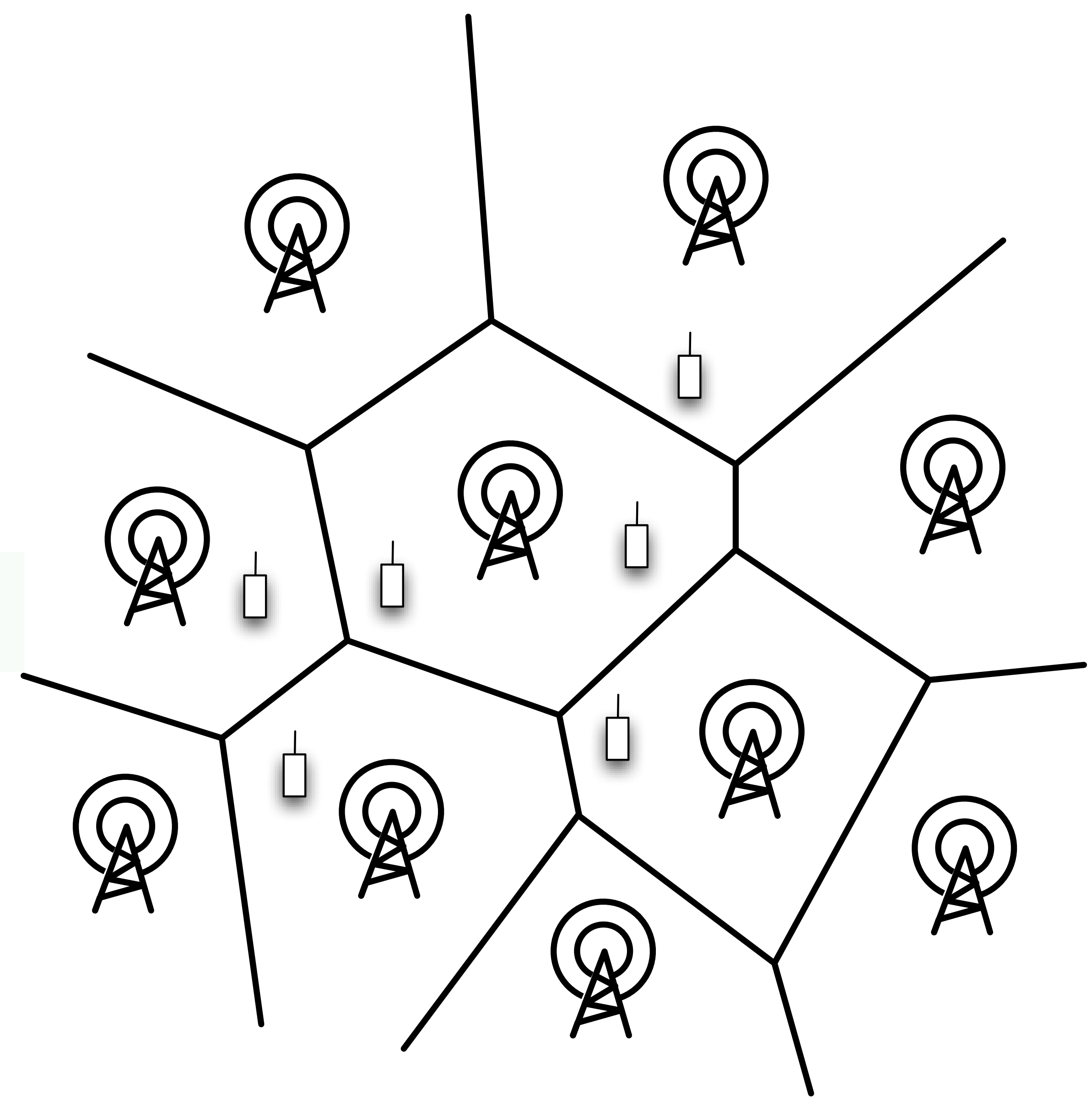
Voronoi Diagrams
Definition (I)

- **Best station problem**
  - leads to Voronoi diagrams
- **Distance measure:**
  - Euclidean norm
  - = L_2-Norm
  - $p=(p_1,p_2)$, $q=(q_1,q_2) \in \mathbf{R}^2$

$$|p,q| := ||p,q||_2 := \sqrt{(p_1-q_1)^2+(p_2-q_2)^2}\ .$$

- **Biscector B(p,q)**

$$B(p,q) := \{x \in \mathbb{R}^2 \mid |p,x| = |q,x|\}\ .$$



Voronoi Diagrams Definition (II)

- **Bisector**

$$B(p,q) := \{x \in \mathbb{R}^2 \mid |p,x| = |q,x|\} \ .$$

- **Partitions the plane in half-planes D(p,q) and D(q,p)**

$$D(p,q) := \{x \in \mathbb{R}^2 \mid |p,x| < |q,x|\}$$

$$D(q,p) = \{x \in \mathbb{R}^2 \mid |p,x| > |q,x|\}$$

- **For a given point set V defined Voronoi region VR(p, V) of a point v $\in$ V:**

$$\mathbf{VR}(p,V) = \bigcap_{q \in V \setminus \{p\}} D(p,q)$$

Voronoi Diagrams
Definition (III)

- **Voronoi region VR(p,V)**

$$\boldsymbol{VR}(p, V) = \bigcap_{q \in S \setminus \{p\}} D(p, q)$$

- **Voronoi diagram VD(V)**

$$\boldsymbol{VD}(V) := \mathbb{R}^2 \setminus \bigcup_{p \in V} \boldsymbol{VR}(p, V)$$

- **All Voronoi regions are convex**
  - Proof: Exercise
- **Voronoi diagrams consist of lines, rays and dots**

Cellular Networks
Voronoi Diagrams

- **Bisector** $B(p,q) := \{x \in \mathbb{R}^2 \mid |p,x| = |q,x|\}$.
- **Half-plane** $D(p,q) := \{x \in \mathbb{R}^2 \mid |p,x| < |q,x|\}$
- **Voronoi region VR(p,V), p ∈ V ⊆ IR²:**

$$\mathit{VR}(p,V) = \bigcap_{q \in V \setminus \{p\}} D(p,q)$$

- **Voronoi diagram VD(V), V ⊆ IR²:**

$$\mathit{VD}(V) := \mathbb{R}^2 \setminus \bigcup_{p \in V} \mathit{VR}(p,V)$$

Voronoi-Diagrams Structure

- **Voronoi diagrams consist of lines, rays and dots**
  - Consider neighboring Voronoi regions VR (p, V) and VR (q, V)
  - Then every point of the common border in the bisector B (p, q) are, because

$$\overline{VR(p,V)} \cap \overline{VR(q,V)} \subseteq \overline{D(p,q)} \cap \overline{D(q,p)} = B(p,q)$$

- **Every Voronoi region is convex and +**
- **Edge pieces consist of a finite number of straight line segments**

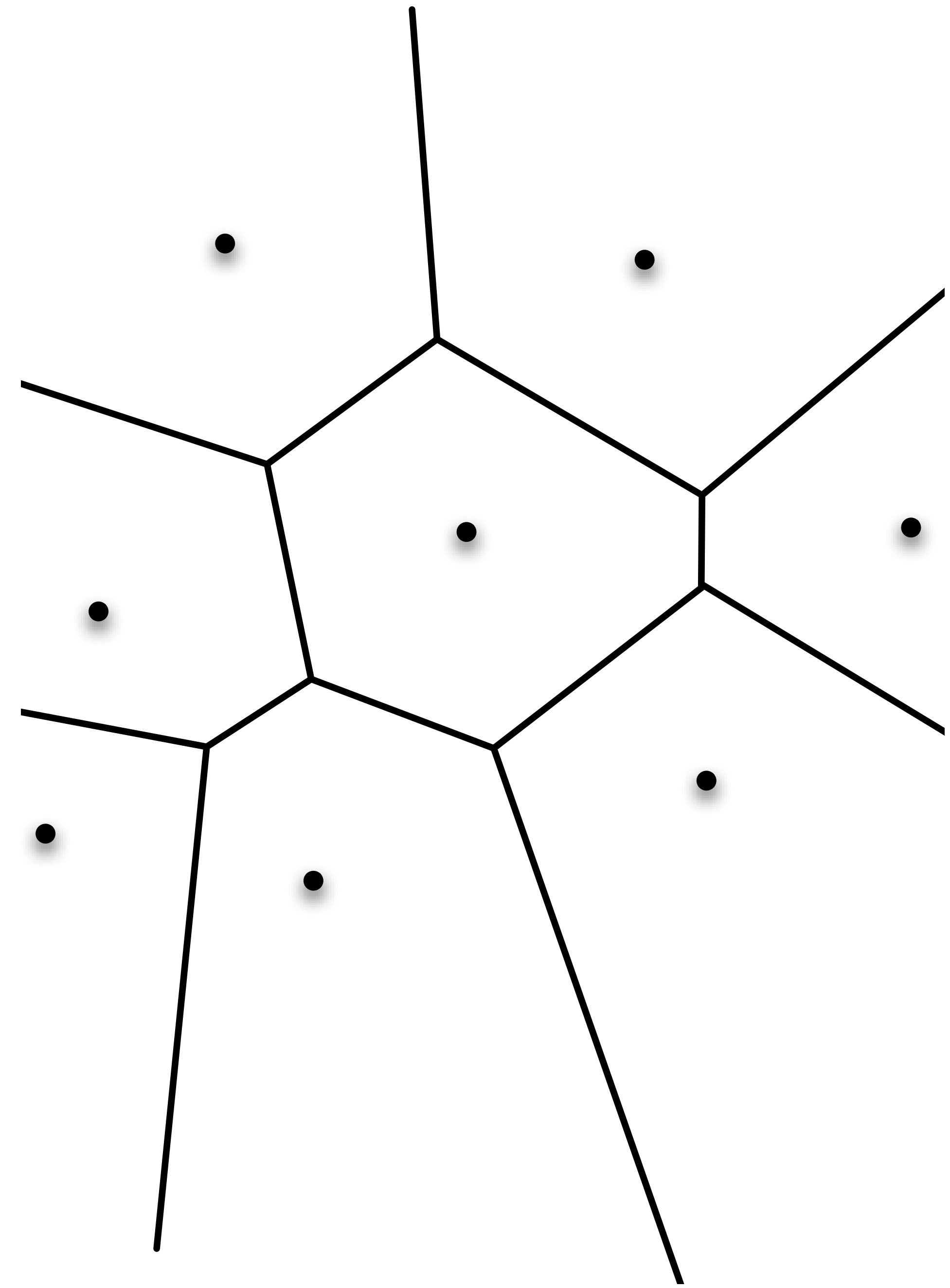
⇒ a Voronoi-Diagram is a geometric graph

Geometric Graph

- **Geometric realization of an undirected graph in R^2**
- **Nodes are mapped to points**
- **Edges are mapped to simple paths**
- **No intersections between two simple paths**
  - planar graph
- **connected component**
  - Maximal subgraph by each node has a path to every other node

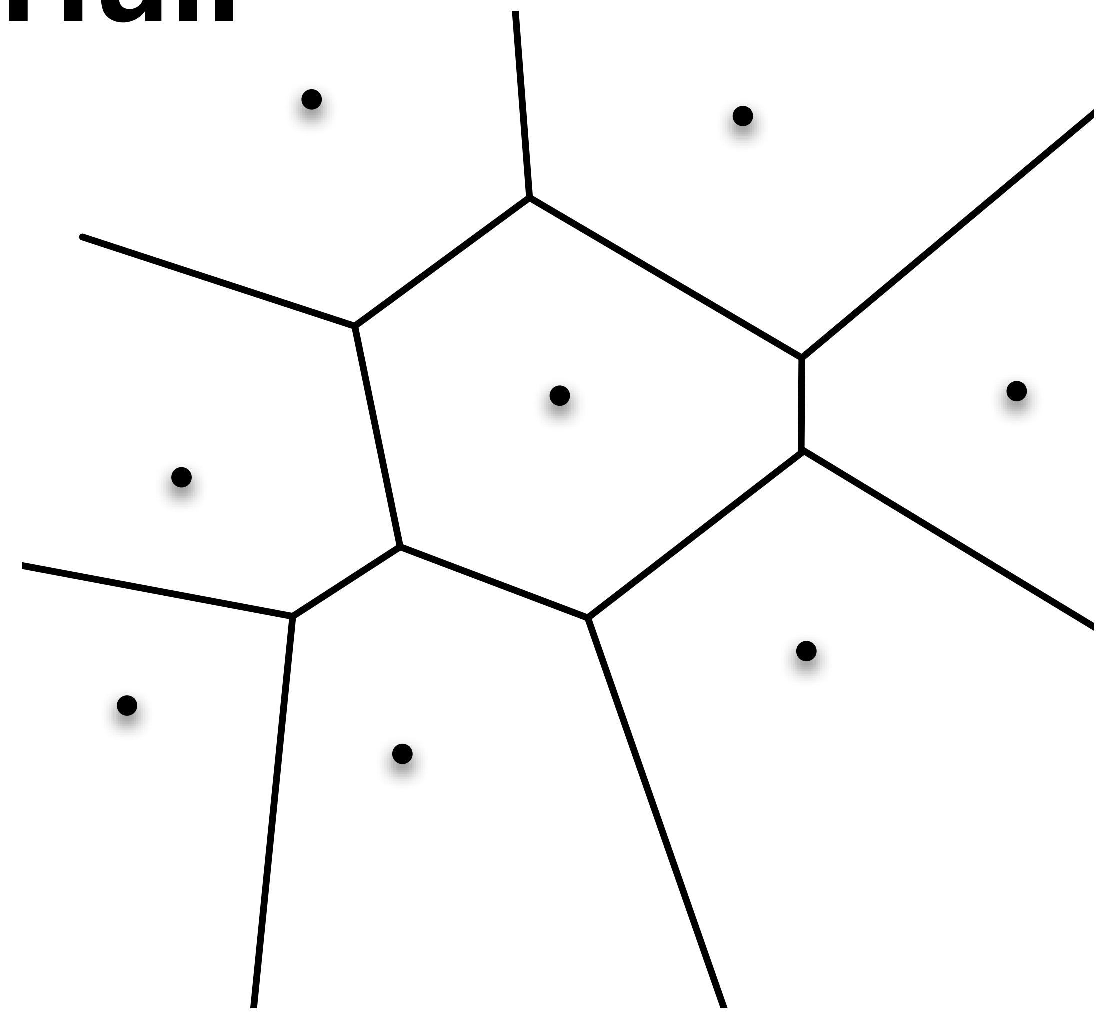
Circle Lemma

- **Lemma**
  - Let x be a point in the plane and let C(x) be an circle expanding with center x. Then for the Voronoi diagram of a point set V:
    1. C(x) first meets only on a point $p \in V$ exactly when x lies in the Voronoi region of p.
    2. C(x) first meets only two points $p, q \in V$ exactly when x lies on a Voronoi edge between the regions of p and q
    3. C(x) first meets exactly at the points p_1, .., p_k, with $k \geq 3$ if and only if x is a Voronoi node at which the adjacent region of p, .., p_k.



Convex Hull

- **Voronoi diagrams are closely related to the convex hull CH (V) of a point set V**



$$CH(V) := \bigcap_{K \supseteq V : K \text{ konvex}} K$$

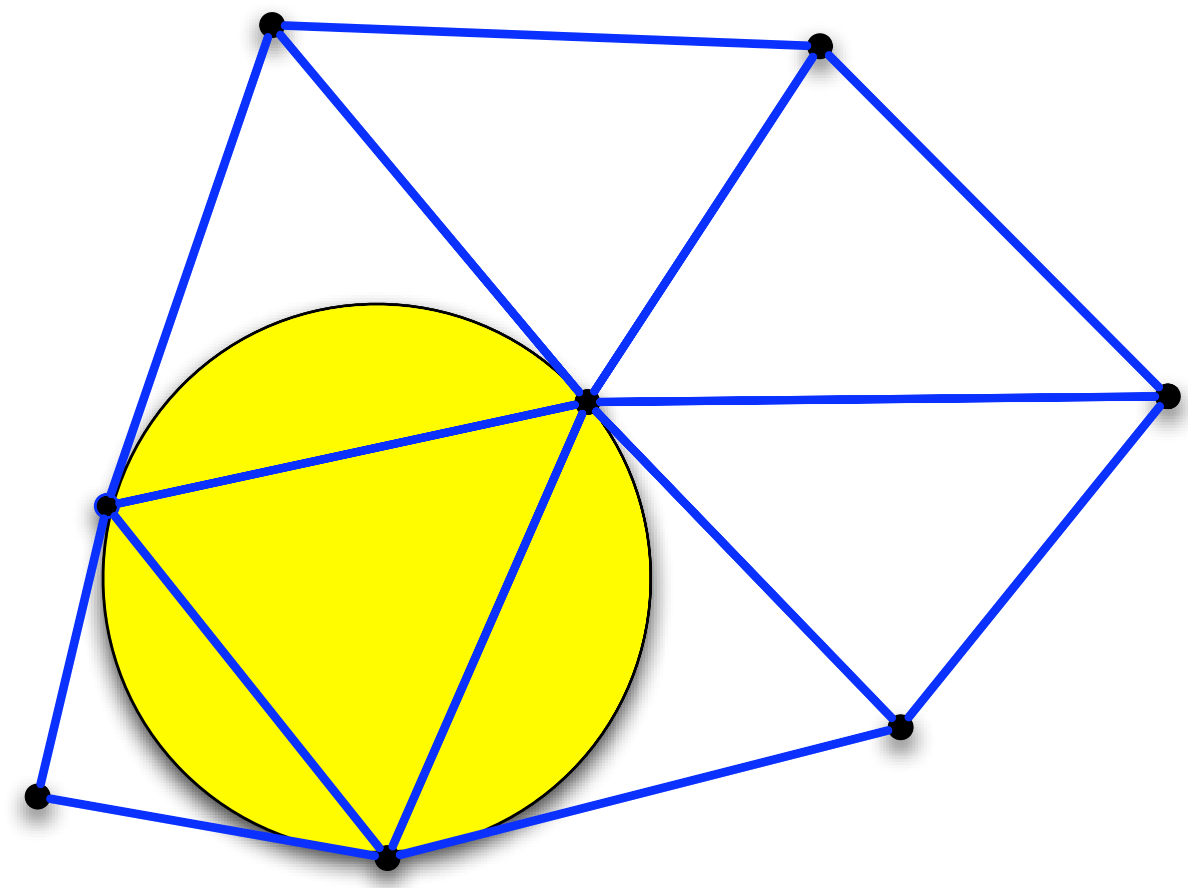
- **Lemma**
  - A point p ∈ V has exactly one unbounded Voronoi region, then, if he is on the edge of the convex hull of V.

Dual Graphs

- **The dual graph G * of a geometric graph G (on a spherical surface) is obtained by proceeding as follows**
  - Inside each face F of G, one chooses a point p^*_F. These points are the nodes of G*.
  - For each edge e of G with an adjacent area F and F' connect p^*_F with $p^*_{F'}$ by an edge e* which intersects only with e and no other edge.

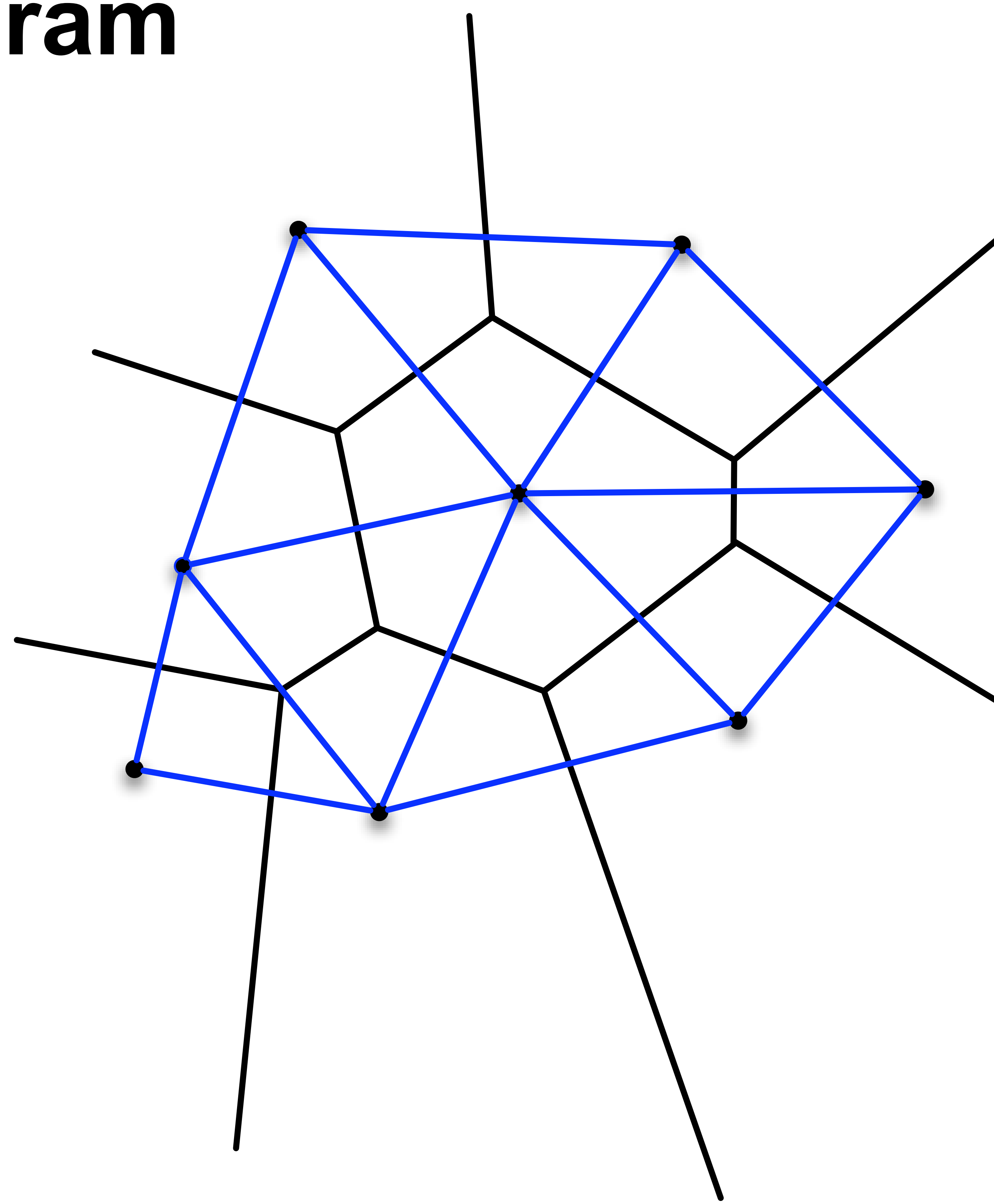
Delaunay Triangulation

- **Definition**
  - For points in general position a Delaunay triangulation consists of triangles that satisfy the radius condition:
    - The perimeter of each triangle does not contain other nodes
- **General position:**
  - No three points are on a straight line
  - No four points on an arc
  - This can be obtained by moving each point to an infinitesimally small vector in random direction



Delaunay from Voronoi-Diagram

- **Alternative representation of the Delaunay triangulation**
  - Given set V and Voronoi-Diagram VD(V)
  - If for p,q∈V the regions VR(p,V) and VR(q,V) share an edge then add (p,q) to the set of edges of the Delaunay triangulation



Delaunay from Voronoi-Diagram

- **Lemma**
  - The Dealunay triangulation is a geometric realization of the dual graph of the Voronoi diagram.
- **Since V in general position, i.e., No four points on an arc**
  - No Voronoi point has more than three edges (because of circle lemma)
  - Thus each face of the dual graph only three adjacent edges
  - $\Rightarrow$ Delaunay triangulation consists only of triangles
- Circle lemma **implies Delaunay perimeter property**

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