

10 Log-Space Reductions

Transitive Closure of Log-Space-Reductions

$$L_1 \leq_m^{\log} L_2$$

- function $f: \Sigma_1^* \rightarrow \Sigma_2^*$

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

f can be computed in log. space.

Theorem

$$L_1 \leq_m^{\log} L_2 \leq_m^{\log} L_3$$

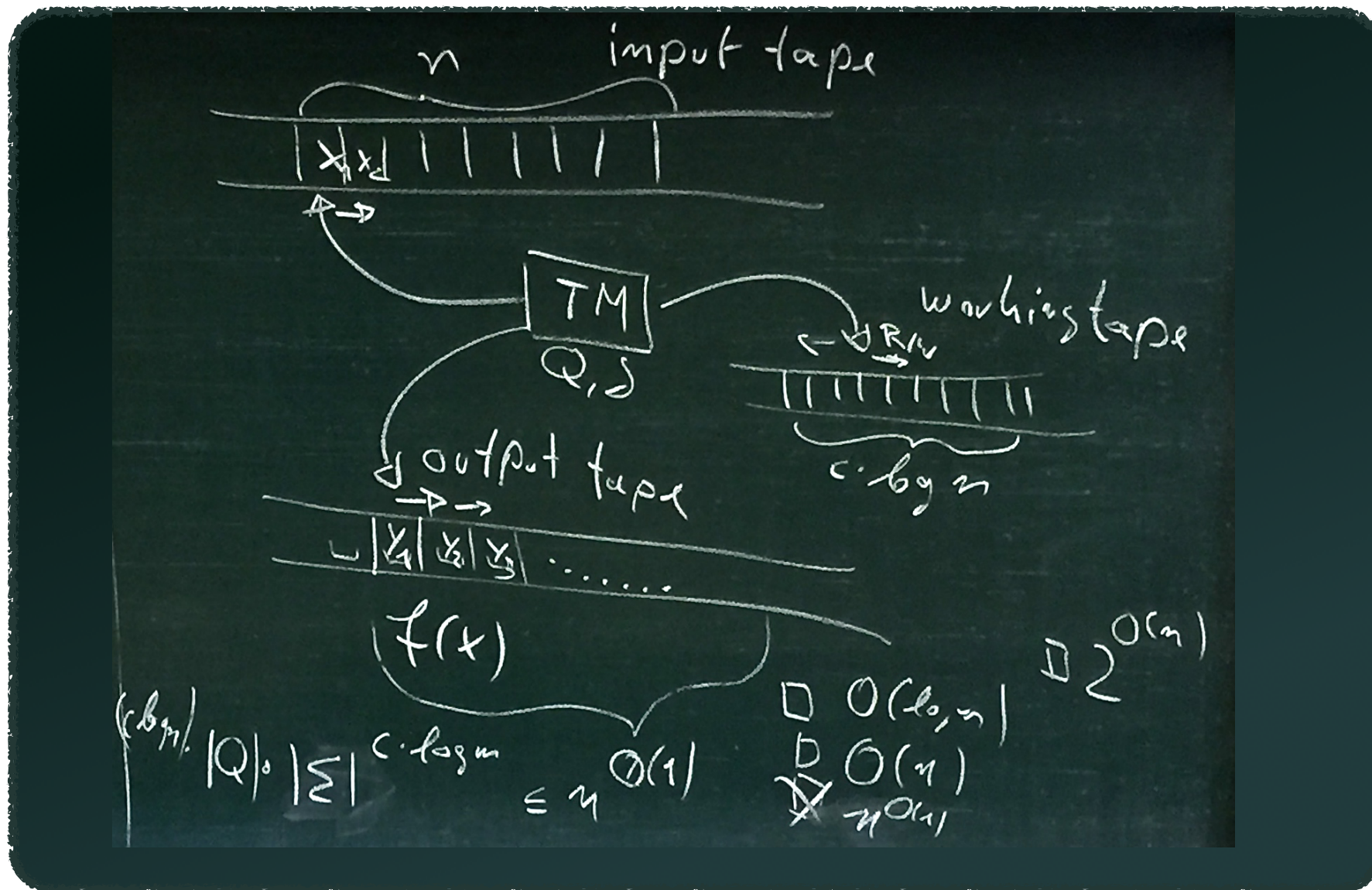
$$\Rightarrow L_1 \leq_m^{\log} L_3$$

$$x \in L_1 \Leftrightarrow f_1(x) \in L_2$$

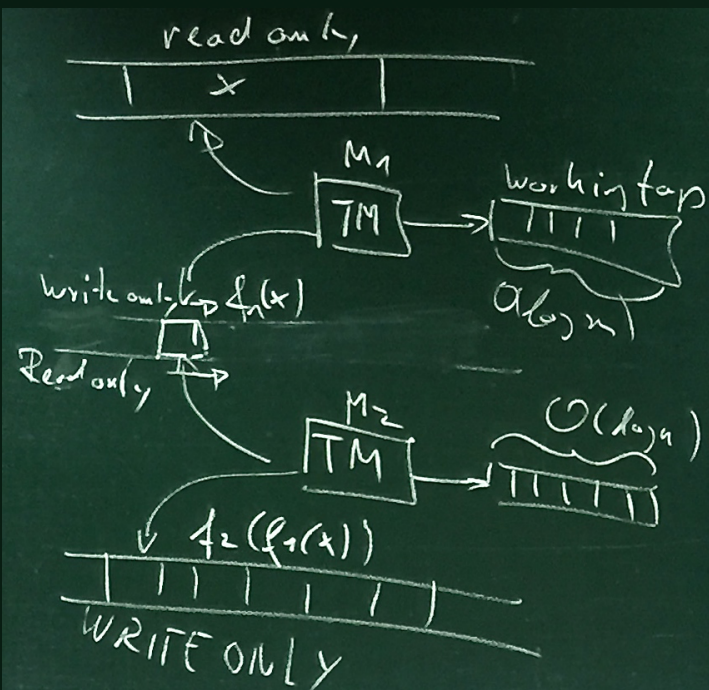
$$y \in L_2 \Leftrightarrow f_2(y) \in L_3$$

$$x \in L_1 \Leftrightarrow f_2(f_1(x)) \in L_3$$

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$$|f_2(f_1(x))| \leq |f_1(x)|^{c_1} \leq |x|^{c_1 c_2} \checkmark$$

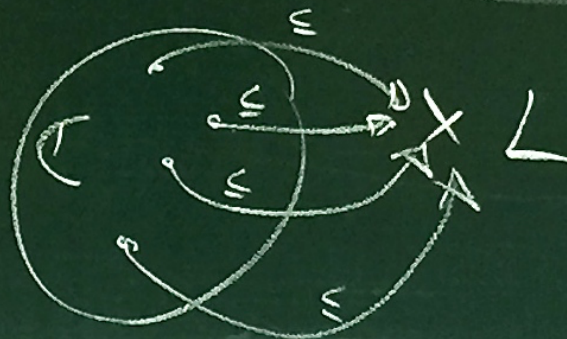
Repeat

1. Simulate $M_1(x)$ until it writes a letter
 2. Simulate $M_2(y)$ until it reads a letter
- until M_1 and M_2 stop.

10 Hardness and Completeness

Hardness

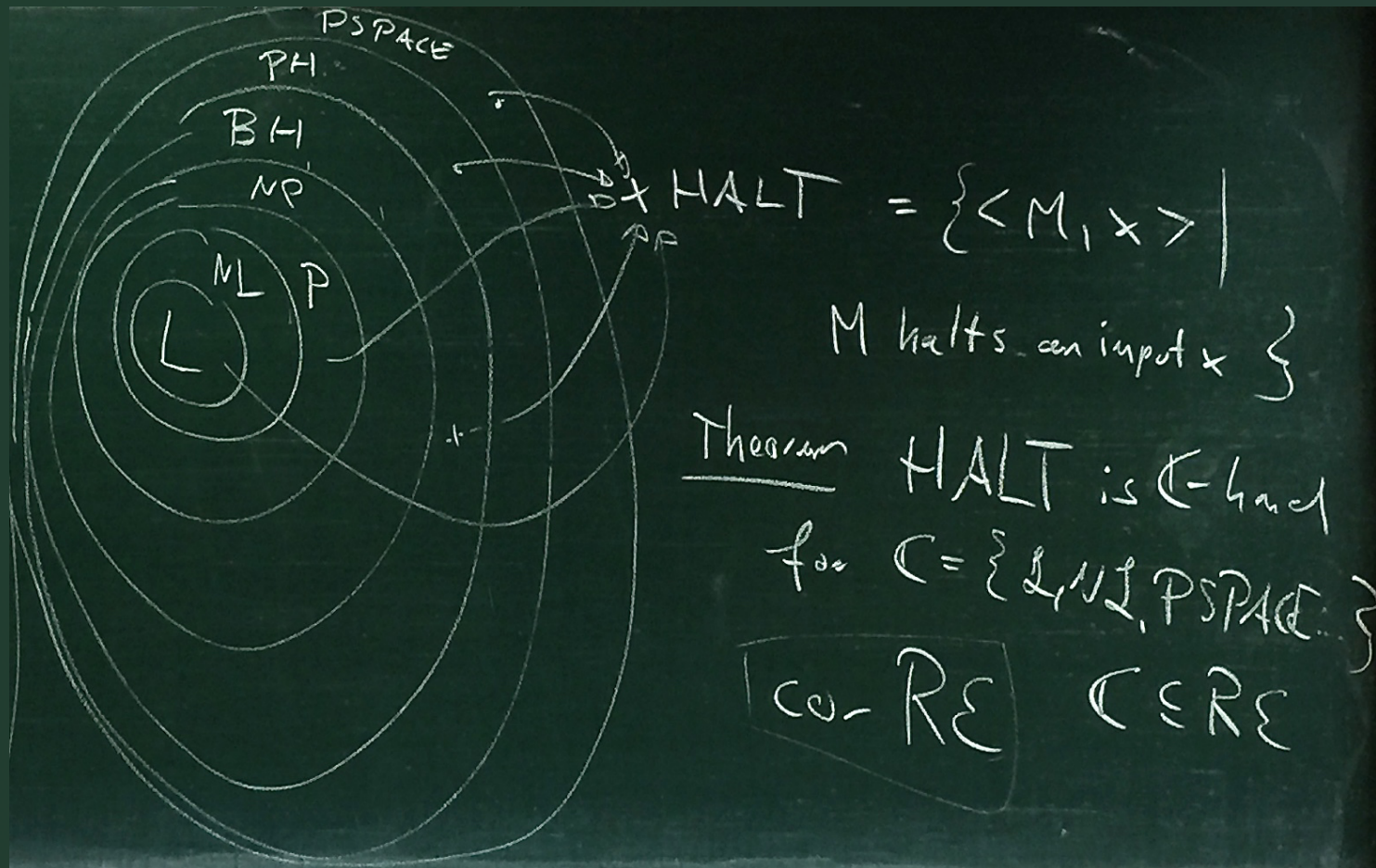
Def. A problem L is
 $\mathbb{C}$ -hard under $\leq_x$ -reducibility
 if $\forall L' \in \mathbb{C}. L' \leq_x L$



10 Hardness and Completeness

Corollary	Class	$\leq_x$
$\mathcal{C}_1$ is closed under $\leq_x$ if $L \in \mathcal{C}_1$ is $\mathcal{C}_2$ -hard then $\mathcal{C}_2 \subseteq \mathcal{C}_1$	NP	$\leq_m^P, \leq_m^{\log}$
	NL, L	$\leq_m^{\log}$
	P	$\leq_m^{\log}$
	PSPACE	$\leq_m^P$
	B+I	$\leq_m^P, \leq_m^{\log}$
	P+I	$\leq_m^P, \leq_m^{\log}$
P is closed under $\leq_m^P$ if $L \in P$ and L is NP-hard $\Rightarrow NP \subseteq P \Rightarrow P = NP$		

10 Hardness and Completeness



10 Hardness and Completeness

Proof $\forall L' \in \mathcal{C}. L' \leq_m^P \text{HALT}$

$L' \in \mathcal{RE} \Rightarrow \exists \text{TM } M' \text{ with } L(M') = L'$

to do: $x \in L' \Leftrightarrow \langle x \rangle \in \text{HALT}$

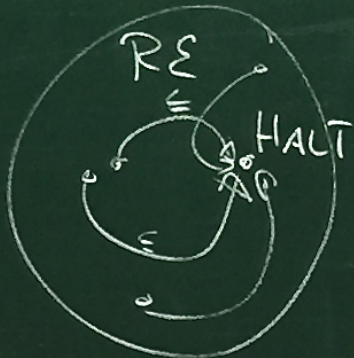
$M'(x) \overset{\text{halts and}}{\text{accepts}} \Leftrightarrow \langle M', x \rangle \in \text{HALT}$

find M such that

$$M(x) = \begin{cases} \text{halt} & \text{if } M'(x) \text{ halts and accepts} \\ \text{crash} & \text{if } M'(x) \text{ crashes} \\ \text{crash} & \text{if } M'(x) \text{ rejects} \end{cases}$$

10 Hardness and Completeness

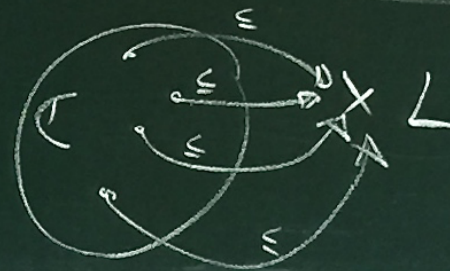
$HALT \in RE$



Def L is C -complete
 1. L is C -hard
 2. $L \in C$

Hardness

Def. A problem L is
 C -hard under $\leq_x$ -reducibility
 if $\forall L' \in C. L' \leq_x L$



10 Hardness and Completeness

Generic Complete Problems

$$L_{\text{TIME}} = \{ \$ M \$ x \$^t \mid \text{DTM } M \text{ accepts } x \text{ in time } t \}$$

$L^* = L_{\text{TIME}}$ is P-complete

1. $L^* \in P$

2. $L^* \in P\text{-hard}$

10 Hardness and Completeness

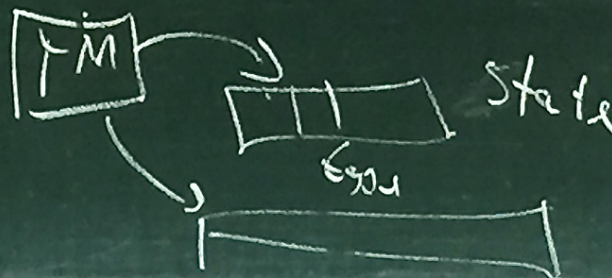
$w = \$ \dots M_1 \ \$ 0101101 \ \$ \$ \$ \$ \$ \$ \$$

t
binary encoding

1. $L_{TIME} \in P$

Simulate $M_1(x)$ for t steps using
the universal TM.

$$\text{time}_{\text{Mon}}(w) \leq (|M| + c) \cdot t \leq |w|^2$$



10 Hardness and Completeness

$$\forall L' \in P. L' \leq_m^{\log} L_{TIME}$$

given M' with $L(M') = L'$
 and $\text{time}_{M'}(x) \leq c_1 |x|^{c_2}$

$$M' \text{ accepts } x \Leftrightarrow f(x) \in L_{TIME}$$

$$f(x) = \$ M' \$ x \$^{c_1 |x|^{c_2}} \in L_{TIME}$$

Compute f in log space

1. WRITE $\$$
2. WRITE " M' " stored in the finite Memory of M'
3. WRITE $\$$
4. COPY input to output
5. compute $c_1 |x|^{c_2}$
 - a) compute binary of $|x|$ $n = \log |x|$
 - b) n^{c_2} space $c_1 \log n$
 - c) multiply with c_1

10 Hardness and Completeness

Generic Complete Problems

$$L_{TIME} = \{ \$M\$ \times \$^t \mid \text{DTM: } M \text{ accepts } x \text{ in time } t \} \quad P\text{-complete}$$

$$L_{NTIME} = \{ \text{" | NTM " " " " } \} \quad \text{NP-complete}$$

$$L_{NSPACE} = \{ \text{" | NTM " " " " } \} \quad \Sigma^P$$

10 Hardness and Completeness

