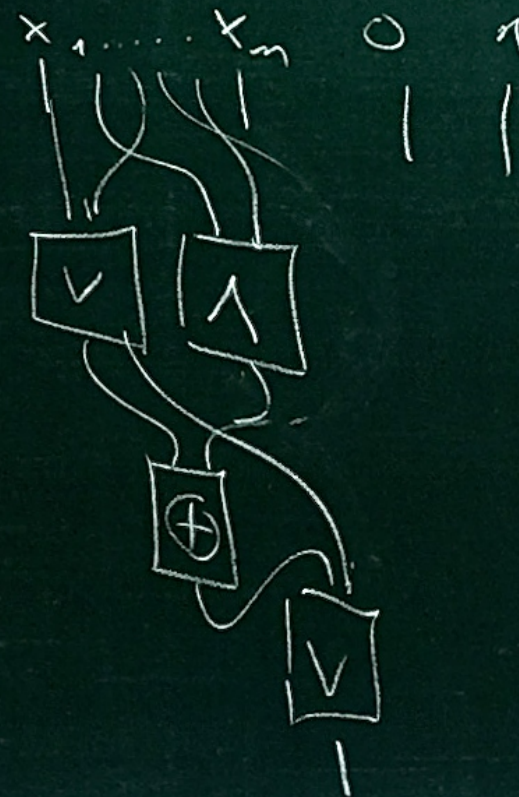


12 Boolean Circuit Size

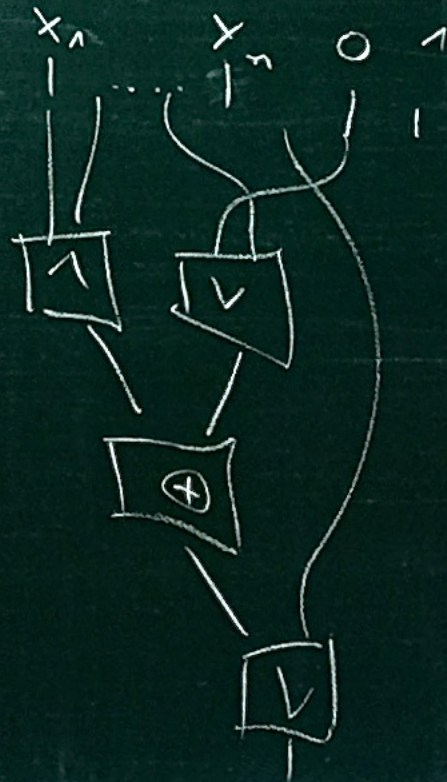
Ω -Circuits

- inputs $x_1, \dots, x_m \in \{0, 1\}$
- constants $0, 1$
- gates $g_1, \dots, g_s \in \Omega \subseteq B_2$
- DAG with k nodes

Boolean Circuits



12 Boolean Circuit Size



Ω -Formulas

- inputs $x_1, \dots, x_n, 0, 1$
- gates $g_1, \dots, g_s \in \Omega \subseteq \mathcal{B}_2$
- (Forest) Tree

12 Boolean Circuit Size

$$\text{Size}_{\Omega}(f) = \min_{\substack{\Omega\text{-circuit } C \\ \text{for } f}} \text{size}(C)$$

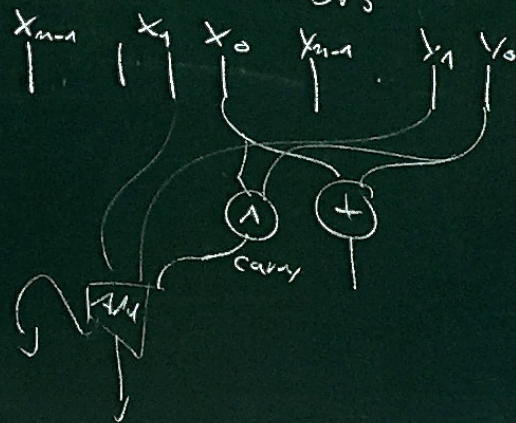
$$\text{Depth}_{\Omega}(f) = \min_{\substack{\Omega\text{-circuit } C \\ \text{for } f}} \text{depth}(C)$$

$$\text{FSize}_{\Omega}(f) = \min_{\substack{\Omega\text{-formula } F \\ \text{for } f}} \text{size}(F)$$

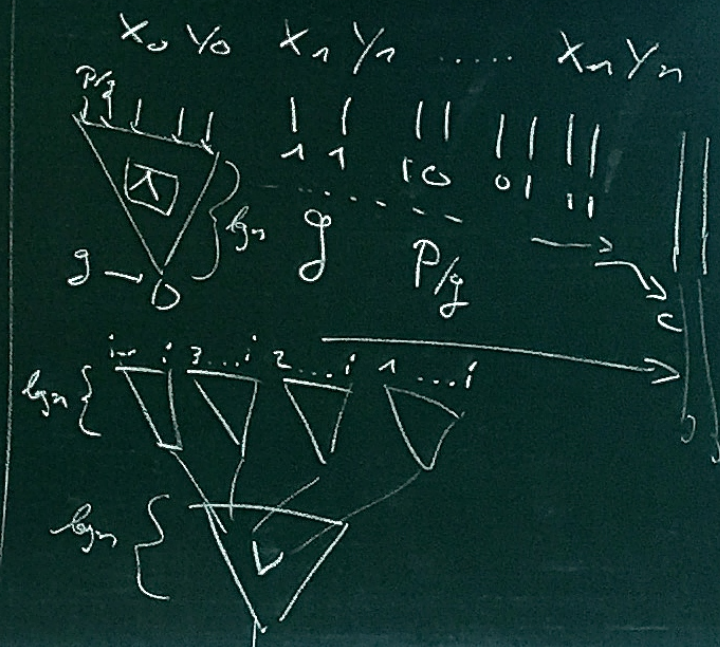
12 Boolean Circuit Size

$$\text{Size}_{\{v, \wedge, \oplus\}}(\text{Addition}_n) = \Theta(n)$$

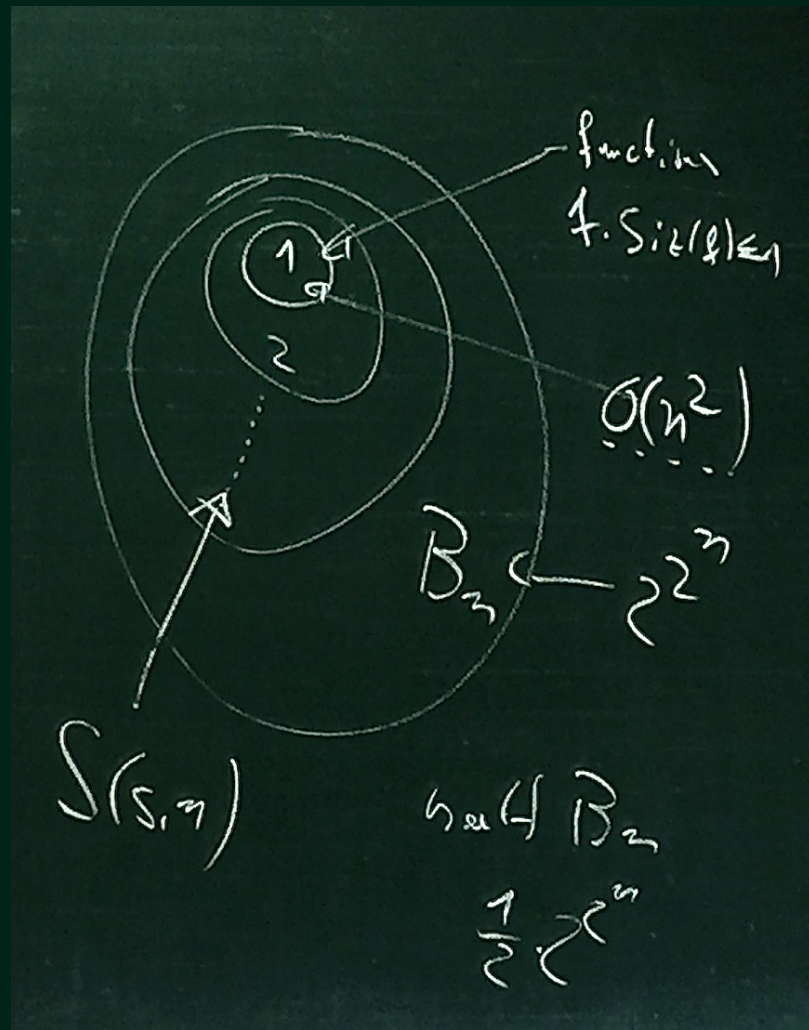
Addition: Add two binary numbers



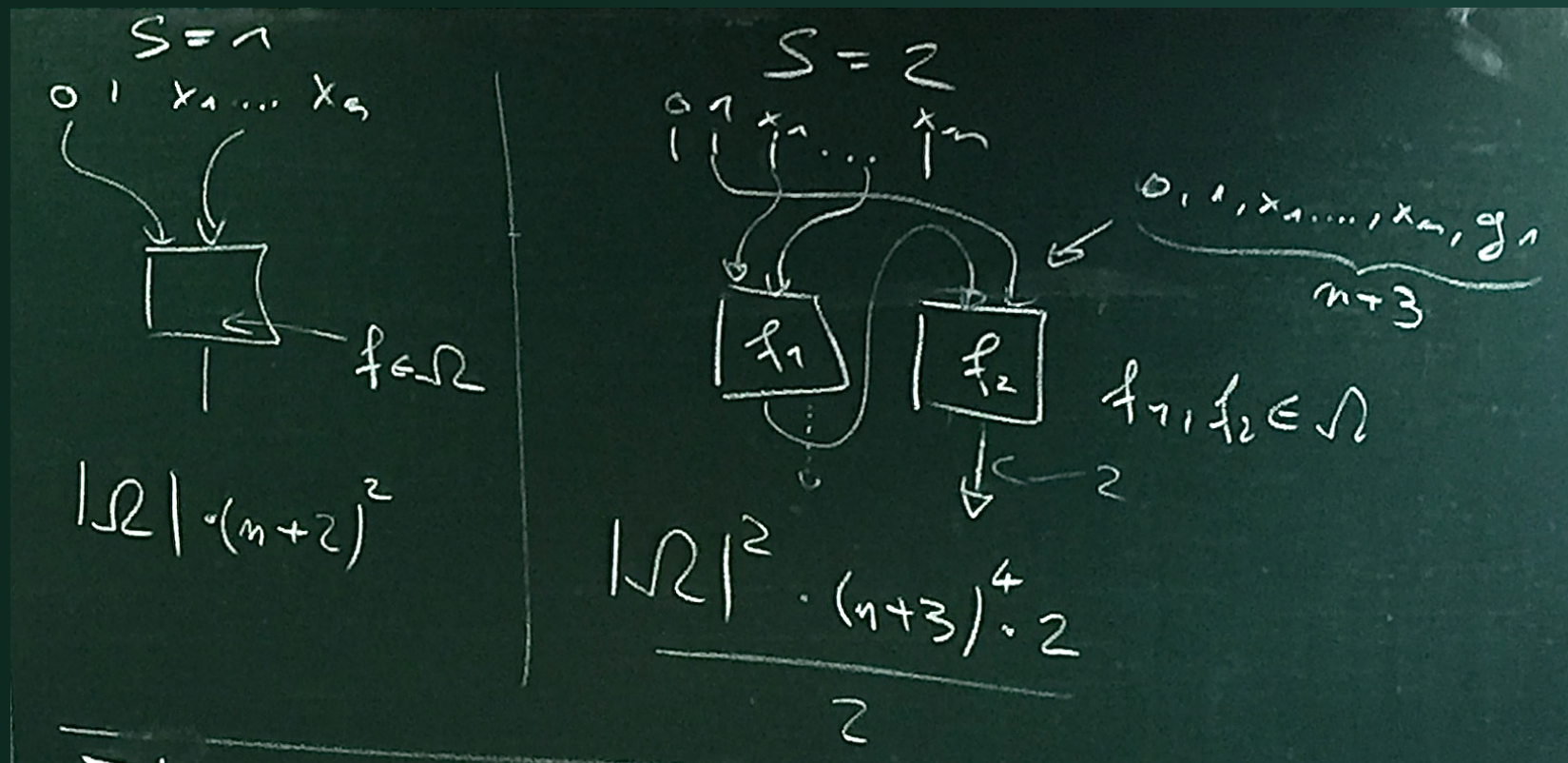
$$\text{Depth}_\wedge(\text{Addition}_n) = \Theta(\lg n)$$



12 Boolean Circuit Size



12 Boolean Circuit Size



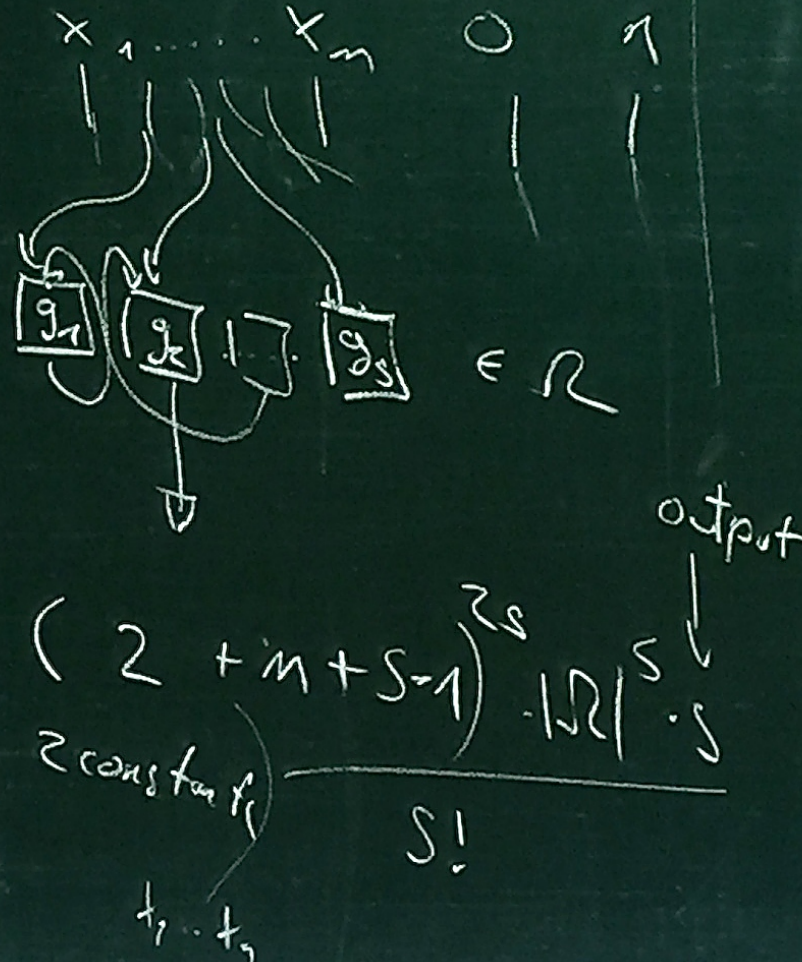
12 Boolean Circuit Size

$$B_n = \{f \mid f: \{0,1\}^n \rightarrow \{0,1\}\}$$

$$|B_n| = 2^{2^n}$$

Lemma The number of $\mathcal{R}$ -circuits of size s is at most

$$S(s, n) = \frac{(s+n+1)^{2s} \cdot |\mathcal{R}|^s}{s!}$$



12 Boolean Circuit Size

$$|B_m| \leq S(s, m)$$

$$2^{2^n} \leq \frac{(1+m+s)^{2s} \cdot |LQ|^s \cdot s}{s!} \quad \Bigg| \log_2$$

$$\Leftrightarrow 2^n \leq \underbrace{\log_2 (1+m+s)^{2s}}_{\leq 2s} + \log_2 |LQ|^s + \log_2 s - \log_2 s!$$

Stirling's formula

$$s! = \Theta \left(\sqrt{2\pi s} \cdot \left(\frac{s}{e} \right)^s \right)$$

$$\log s! = \frac{1}{2} \log s + \frac{1}{2} \log 2\pi + s \log s - s \log e + o(1)$$

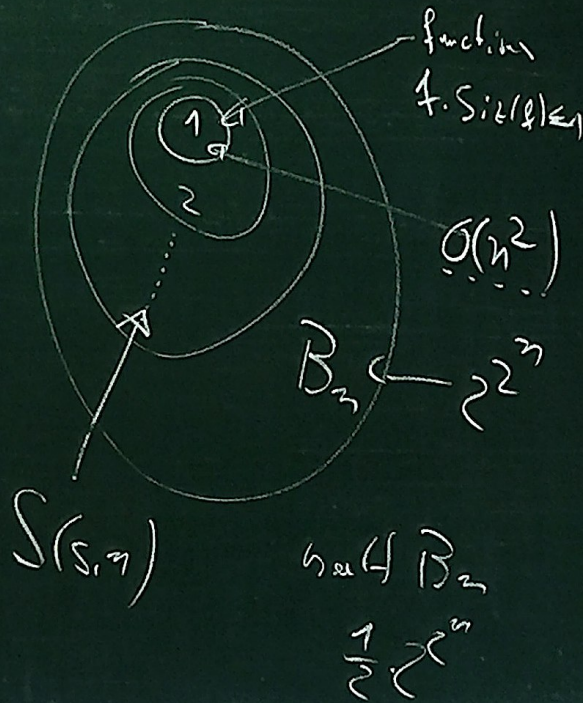
$$\begin{aligned} \log a \cdot b &= \log a + \log b \\ \log a^b &= b \cdot \log a \end{aligned}$$

12 Boolean Circuit Size

$$\begin{aligned}
 & \underbrace{(\log s) + 1} \\
 & 2s \cdot \log 2s + s \cdot \log |Q| + \log s - \\
 & \left(\frac{1}{2} \log s + \frac{1}{2} \log 2\pi + s \log s - s \log e \right) + \Theta(1) \\
 & = s \cdot \log s + s \underbrace{(2 + \log |Q| + \log e)} \\
 & \quad + \frac{1}{2} \log s + \Theta(1)
 \end{aligned}$$

12 Boolean Circuit Size

Theorem For large enough n
 there are functions $f \in \mathcal{B}_n$
 with $\text{size}(f) > \frac{2^n}{n}$



$$2^n \leq S \cdot \log S + c \cdot S$$

$$\Rightarrow S > \frac{2^n}{n}$$

$$S = \frac{2^n}{n} \quad \log S = n - \log n$$

$$\frac{2^n}{n} (n - \log n) + c \frac{2^n}{n}$$

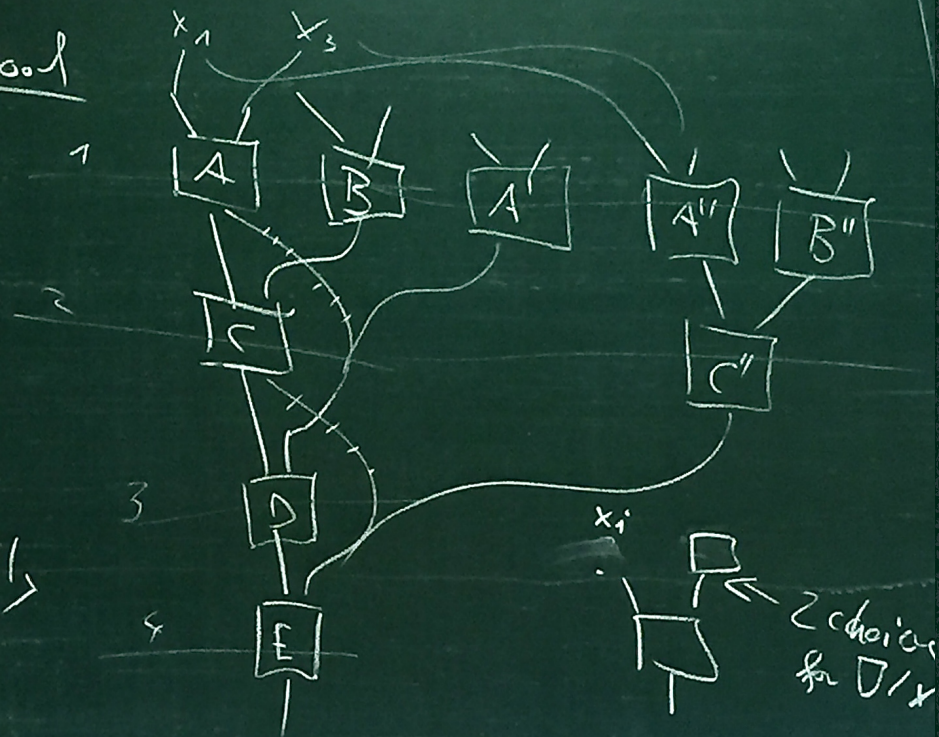
$$2^n \left(1 - \frac{\log n}{n} + \frac{c}{n} \right)$$

12 Boolean Circuit Size

Lemma Every circuit of depth d has a formula of depth d .

- copy gates and parts of circuit s.t. we get a tree with the same functionality

Proof



$$\underbrace{1 \in \mathcal{G}_1} \quad \underbrace{1 \in \mathcal{G}_1, \dots, \mathcal{G}_n}$$

12 Boolean Circuit Size

Fact DNF gives a circuit
of size $O(2^n \cdot n)$

Theorem $\forall f \in \mathcal{B}_n$

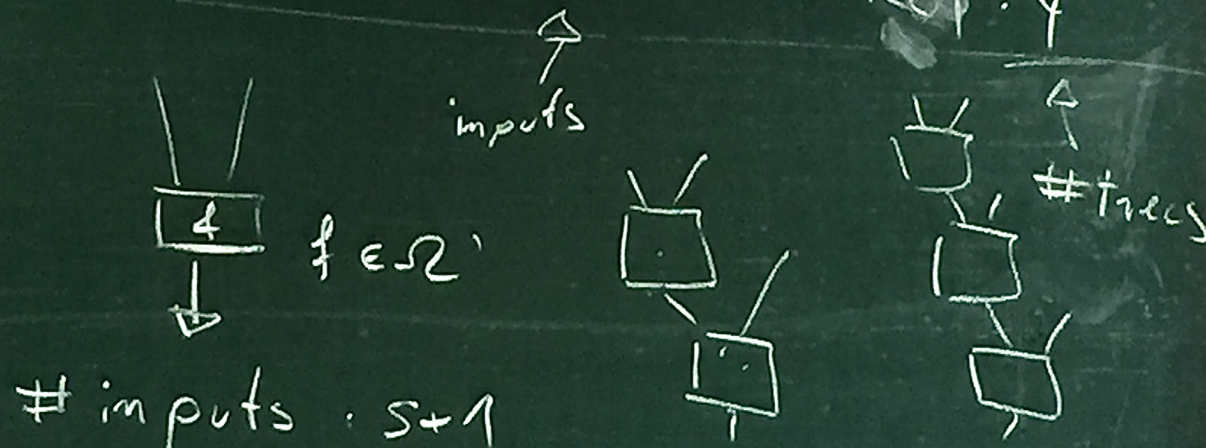
$$\text{Size}_{\mathcal{B}_2}(f) \leq \frac{2^n}{n} + o\left(\frac{2^n}{n}\right)$$

12 Boolean Circuit Size

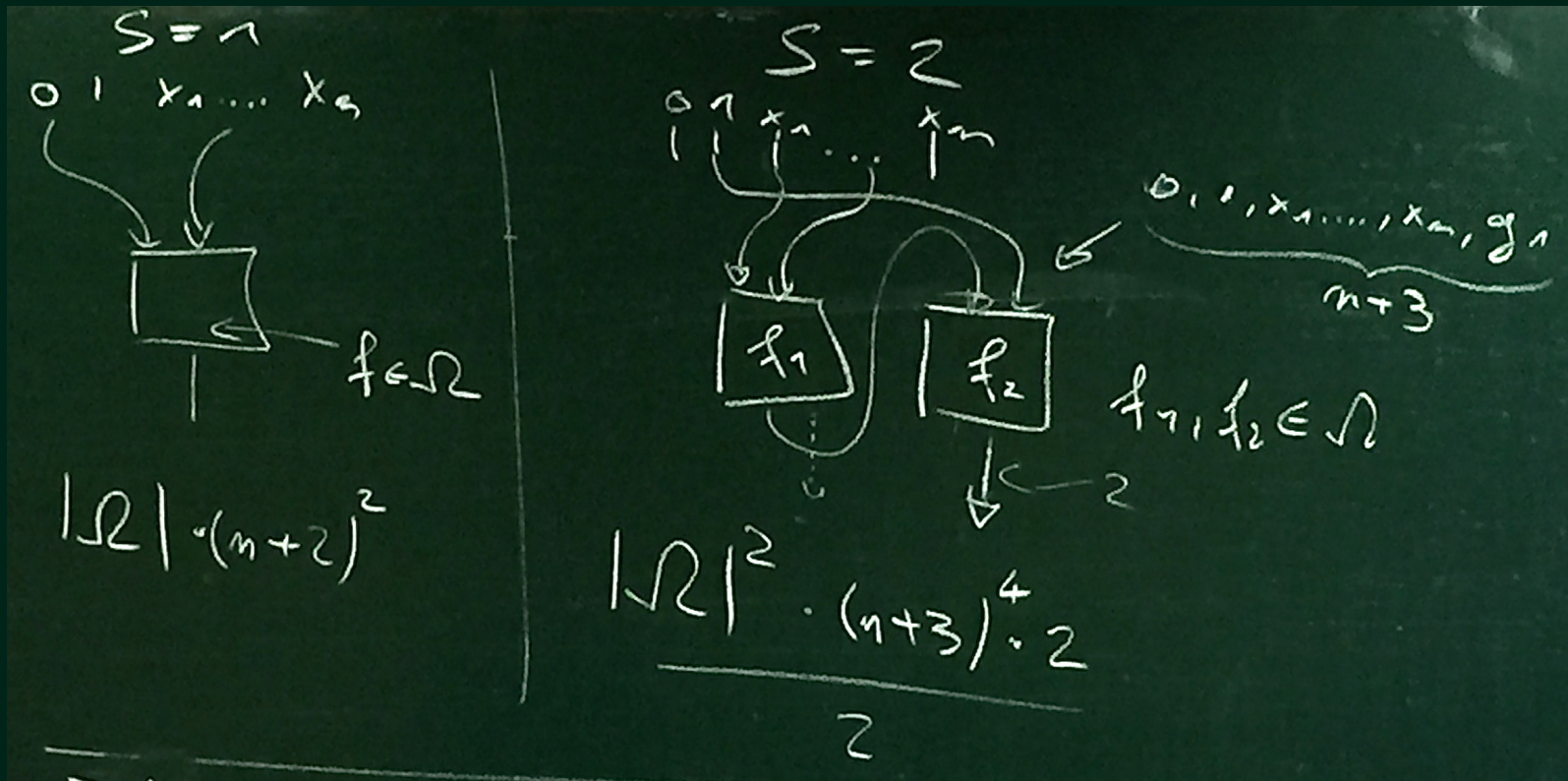
Lemma with Ω a formula of size s

we can describe at most $F(s, n)$ functions $\mathcal{B}_n$

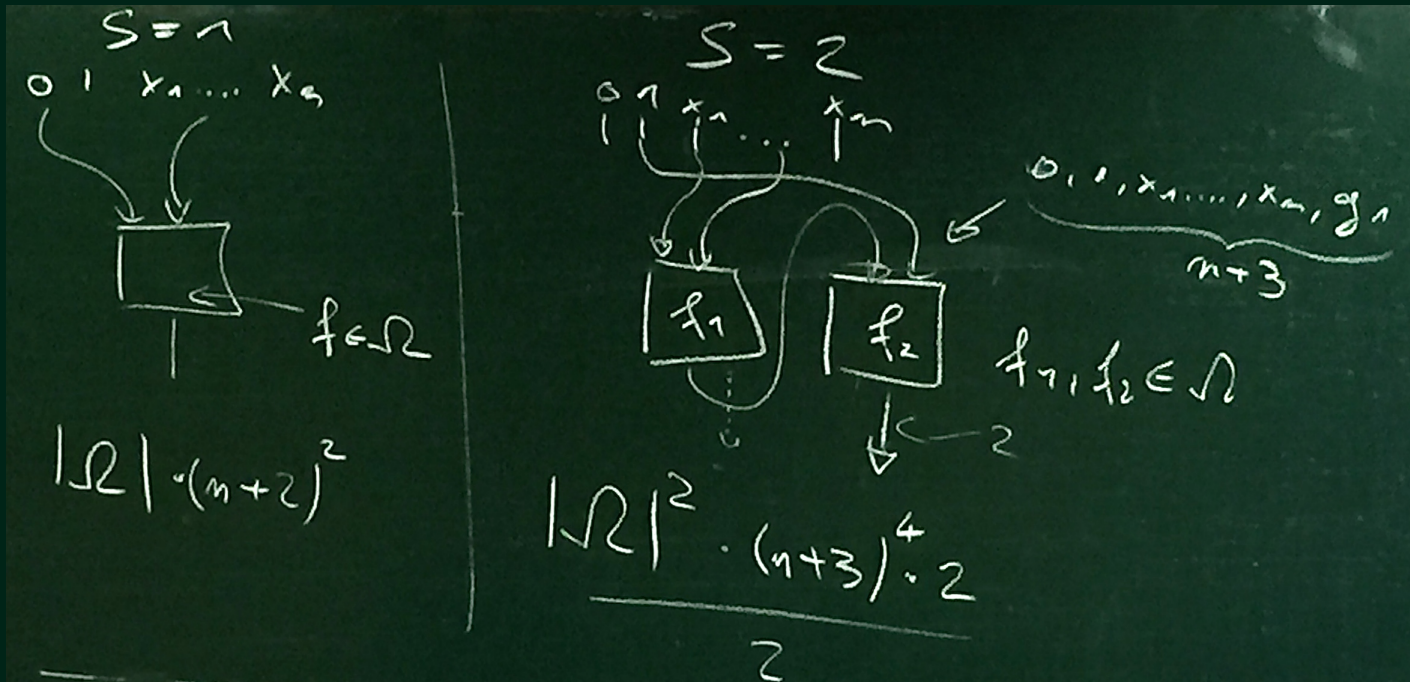
$$F(s, n) = (n+2)^{s+1} \cdot 12^s \cdot 4^s$$



12 Boolean Circuit Size



12 Boolean Circuit Size



Theorem For large enough n
 there are functions $f \in \mathcal{B}_n$
 with $\text{size}(f) \geq \frac{2^n}{n}$