

## Average Complexity

– Expected Time

$x \in \Sigma^*$  :  $\text{time}_M(x) := \# \text{ stops}$   
 until  $M$  computes  
 the result on input  $x$

$$E_{\mu}(\text{time}_M(x))$$

$$\sum_{x \in \Sigma^*} P_{\mu}[X=x] = 1$$

$$x \in \Sigma^* : P_{\mu}[X=x] \in [0,1]$$

Notation:  $u'(x) = P_{\mu}[X=x]$

$$M(x) = \sum_{x' \in x} P_{\mu}[X=x']$$

$\underbrace{\hspace{10em}}_{u'(x)}$

# 20 Average-P, DisNP

"Natural" prob. distr.

- Same prob. for inputs  
 $\hat{=}$  uniform distr.

first coin    1  $\rightarrow$  cont.  
                   0  $\rightarrow$  stop

second coin    0, 1

|               |               |               |                |
|---------------|---------------|---------------|----------------|
| $\epsilon$    | 0             | 00            | 000            |
|               | 1             | 01            | ...            |
| $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ |

$$P[X=x] = \frac{1}{2^{|x|+1}}$$

$x \in \{0,1\}^*$

# 20 Average-P, DisNP

$$\forall n: \quad E \left[ \text{time}_M(x) \right] \leq |x|^c$$

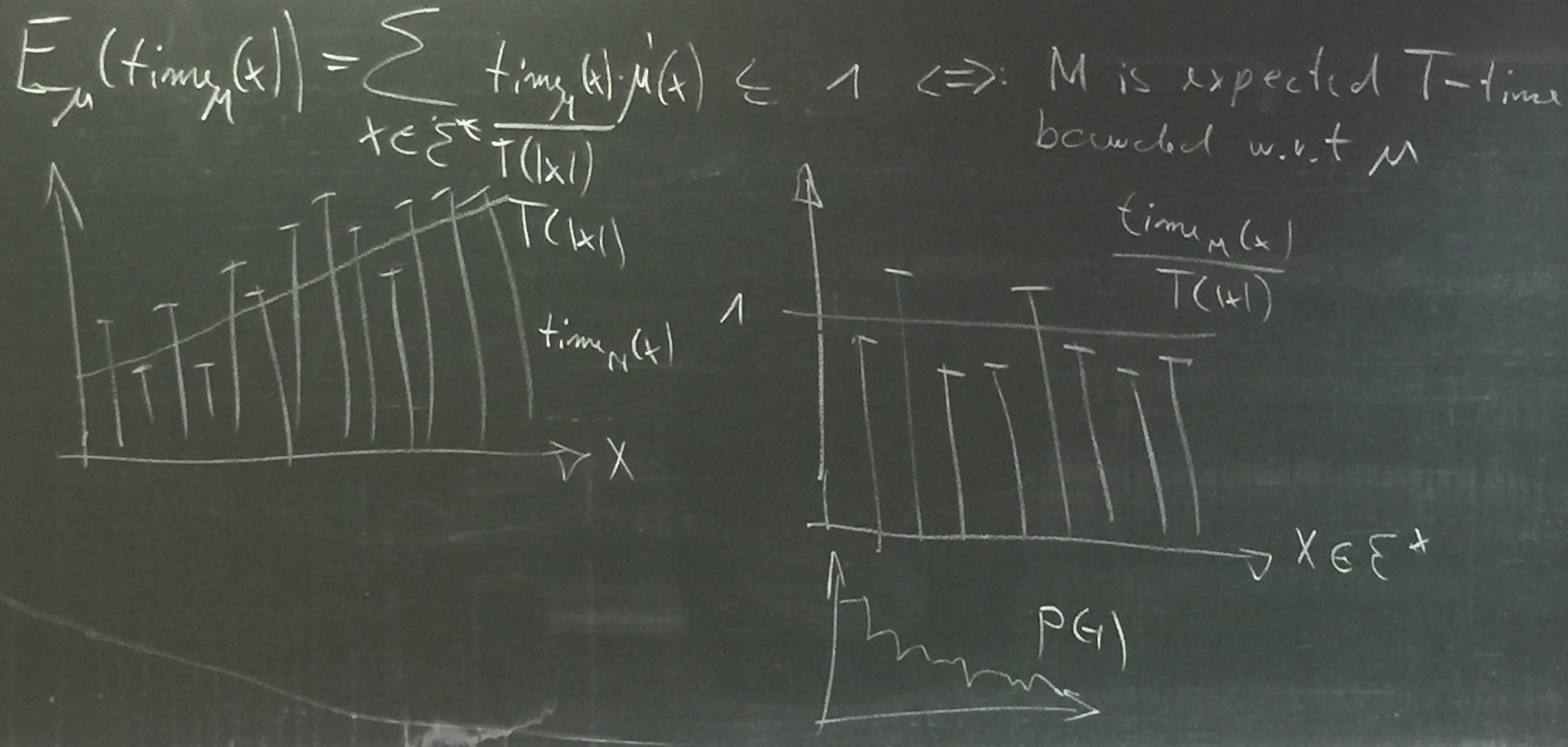
$x \in \Sigma^n$

$$L = \{ 1, \underbrace{11}_{1^2}, \underbrace{1111}_{1^4}, \underbrace{111111}_{1^6}, \dots \}$$

binary language

$\Rightarrow$  worst case measure  
 for tally languages

# 20 Average-P, DisNP



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$$(f, M) \in E-T$$

$$\Leftrightarrow \sum_{x \in \Sigma^*} \frac{f(x) \cdot M'(x)}{T(x)} \leq 1$$

$$(f, M) \in E-Pol$$

$$\Leftrightarrow \exists c \in \mathbb{N}$$

$$\sum_{x \in \Sigma^*} \frac{f(x)}{c|x|^c} \cdot M'(x) \leq 1$$

Simulation, reduction

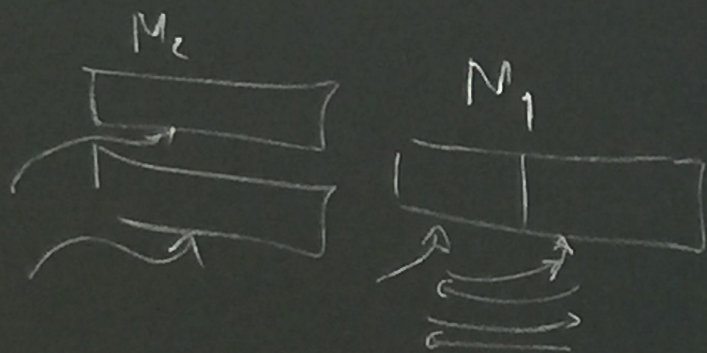
1-tape TM  $M_1$ , 2-tape TM  $M_2$

$$(\text{time}_{M_2}(M)) \in E-Pol$$

$$\stackrel{?}{\Rightarrow} (\text{time}_{M_1}(M)) \in E-Pol$$

# 20 Average-P, DisNP

$$\text{time}_{M_1}(x) \leq \text{time}_{M_2}(x)^2$$



$$\Sigma = \{0, 1\}$$

$$M'(x) = g(n) \cdot \frac{1}{\sum n}$$

$$\sum_{n=0}^{\infty} g(n) = 1$$

$$\text{time}_{M_2}(x) = \begin{cases} |x|, & x \notin 0^* \\ 2^{|x|}, & x \in 0^* \end{cases}$$

$$\sum_{x \in \Sigma^*} \frac{\text{time}_{M_2}(x)}{2 \cdot |x|} \cdot M'(x)$$

$$= \sum_{x \in \Sigma^*} \frac{|x|}{2|x|} \cdot M'(x) + \dots$$

# 20 Average-P, DisNP

$$\text{time}_{M_1}(x) = \begin{cases} |x|^2 & x \notin 0^* \\ 2^{|x|} & x \in 0^* \end{cases}$$

$$\sum_{x \in \Sigma^*} \frac{\text{time}_{M_1}(x)}{c_1 |x|^{c_2}} \cdot M'(x) = \sum_{x \in \Sigma^*} \underbrace{\frac{|x|^2}{c_1 |x|^{c_2}}}_{= o(1)} \cdot M'(x) + \sum_{n=0}^{\infty} \frac{2^{\sqrt{|x|}}}{c_1 |x|^{c_2}} \frac{1}{2^{|x|}} g(n)$$

$$g(n) = \frac{1}{\pi^2} \frac{6}{\pi^2} \left| \sum_{m=1}^{\infty} g(m) = 1 \right.$$

$$\Rightarrow \sum_{n=0}^{\infty} g(n) \cdot \frac{2^{\sqrt{|x|}}}{c_1 |x|^{c_2}} \rightarrow \infty \quad \frac{o(1)}{\frac{1}{\pi^2}} \in \omega(|x|^c)$$

# 20 Average-P, DisNP

$$\begin{aligned}
 &= \sum_{x \in \Sigma^* \setminus \{0^*\}} \frac{|x|}{2^{|x|}} M'(x) + \sum_{\substack{m=0 \\ x=0^n}}^{\infty} \frac{2^{|x|}}{2^{|x|}} M'(x) \\
 & \qquad \qquad \qquad \underbrace{\qquad \qquad \qquad}_{g(n) \cdot \frac{1}{2^{|x|}}} \\
 &= \sum_{x \in \Sigma^*} \frac{1}{2^{|x|}} M'(x) + \sum_{n=0}^{\infty} \frac{g(n)}{2 \cdot n} \\
 & \leq \frac{1}{2} + \frac{1}{2} \leq 1
 \end{aligned}$$



# 20 Average-P, DisNP

$$(f, \mu) \in E-T$$

$$\Leftrightarrow \sum_{x \in \Sigma^*} \frac{f(x) \cdot \mu'(x)}{T(|x|)} \leq 1$$

$$(f, \mu) \in Av-T$$

$$\Leftrightarrow \sum \frac{T^{-1}(f(x))}{|x|} \mu'(x) \leq 1$$

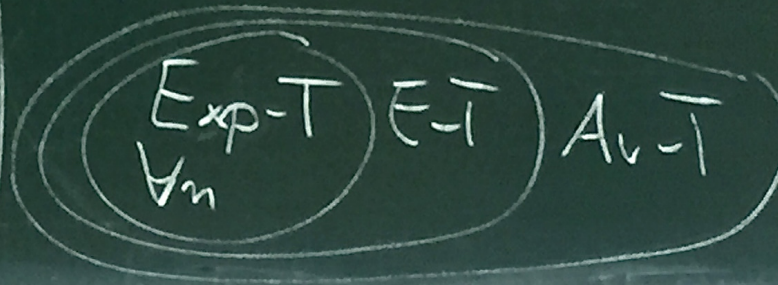
$$(f, \mu) \in E-Pol$$

$$\Leftrightarrow \exists c \in \mathbb{N}$$

$$\sum_{x \in \Sigma^*} \frac{f(x)}{c^{|x|}} \mu'(x) \leq 1$$

if  $T$  is monotone

$$(f, \mu) \in E-T \Rightarrow (f, \mu) \in Av-T$$



$$\frac{T^{-1}(f(x))}{|x|} = 1$$

$$\Leftrightarrow T^{-1}(f(x)) = |x|$$

$$\Leftrightarrow f(x) = T(|x|)$$

$$(f, \mu) \in \text{Av-P}_0$$

$$\Leftrightarrow \exists c_1, c_2 \cdot \sum_{x \in \Sigma^*} \frac{c_2 (f(x))^{c_1}}{|x|} \leq 1$$

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$$\sum_x \frac{f(x)^{c_1} \cdot c_1}{|x|} \mu'(x) \leq 1$$

Lemma  $(f, \mu) \in \text{Av-P}_0$

$$\Rightarrow (f^2, \mu) \in \text{Av-P}_0$$

# 20 Average-P, DisNP

$$Av-P = \{(L, \mu) \mid \exists \text{DTM } M, (\text{time}_{M, \mu}) \in Av-Pol\}$$

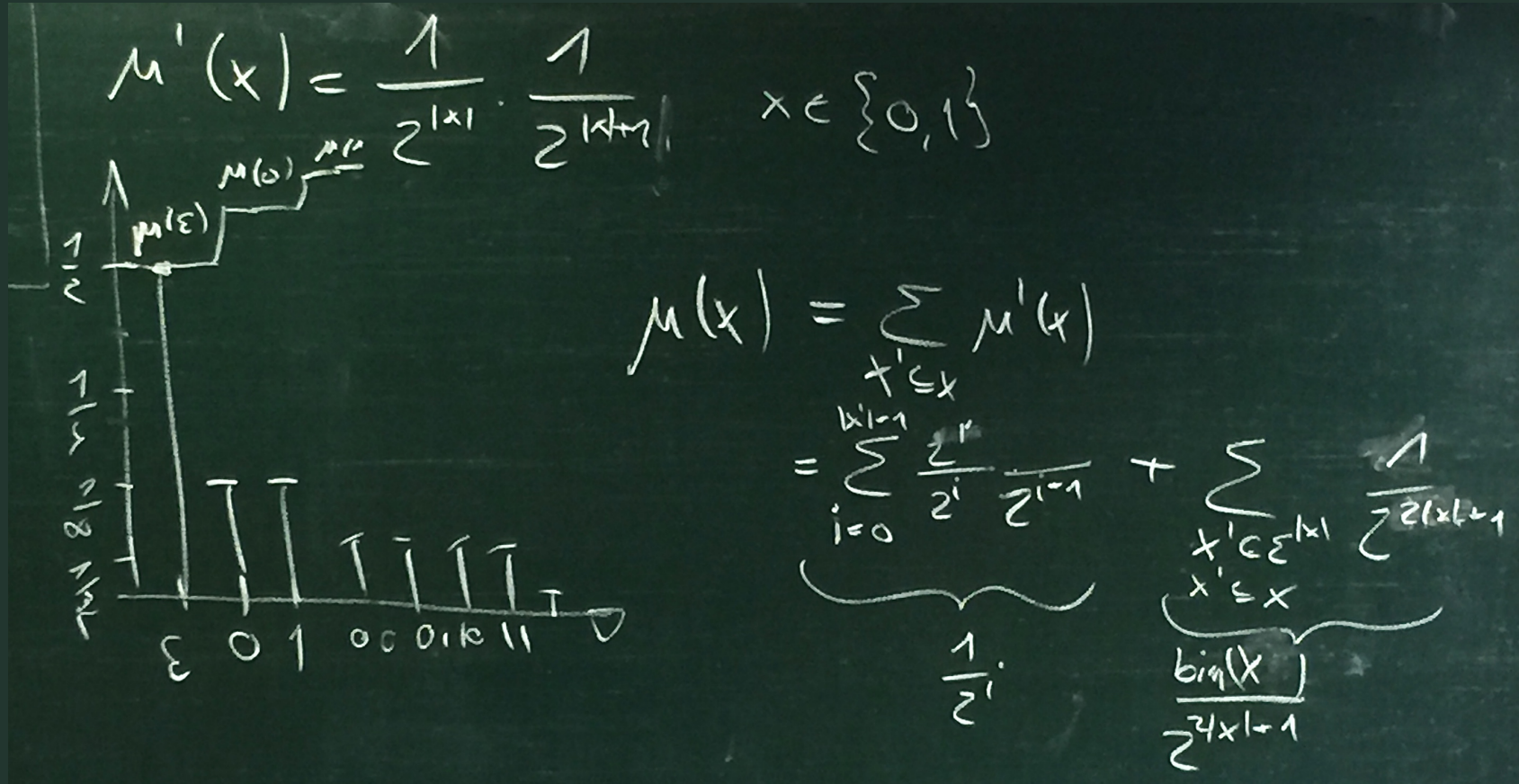
Def  $\mu \in \text{POL-computable}$

- $\exists \text{DTM } M$ , with input  $x$ ,  $1^m$
- $M$  outputs  $\mu(x)$  up to  $m$  digits
- $M$  is pol time bounded in  $|x|+m$

Def  $\mu \in \text{POL-Sampleable}$

- $\exists \text{PTM}$  with no input
- outputs  $x$  in time  $\leq |x|^{c_x}$
- with prob.  $\mu'(x)$

# 20 Average-P, DisNP



# 20 Average-P, DisNP

$SAT \in NP-C$

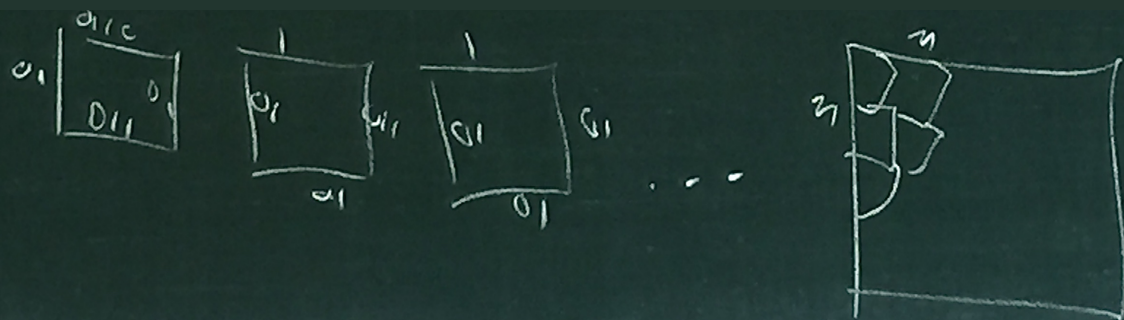
$SAT \in P \Rightarrow P = NP$

$(RT_{1/m_0}) \in DisNP-C$

$(RT_{1/m_{avg}}) \in Av-P \Rightarrow DisNP \in Av-P$

↑  
natural

NP × POL-compatible  
suppleable



# 20 Average-P, DisNP

$$(L_1, \mu_1) \leq_{\text{av-pol}} (L_2, \mu_2)$$

$\uparrow$   
 natural

- $L_1 \leq_{\text{pol}} L_2$
- $\mu?$

$$\mu_2(x) = \mu_1(R(x))$$

$R$  injective

$$(L_2, \mu_2) \in \text{AV-P}$$

$$\Rightarrow (L_1, \mu_1) \in \text{AV-P}$$

$$\text{SAT} \leq_{\text{pol}} \text{Hamilton}$$

