

5 Closures and Canonical Computation Trees

Closure Properties

1. Complement $C \subseteq 2^{\Sigma^*}$

$$\text{co-}C := \{L \mid \bar{L} \in C\}$$

$$X_L(x) = \begin{cases} 0, & x \notin L \\ 1, & x \in L \end{cases}$$

$$X_{\bar{L}}(x) = \neg X_L(x) = \begin{cases} 1, & x \notin L \\ 0, & x \in L \end{cases}$$

C is closed under complement

$$\text{if } C = \text{co-}C \iff L \in C \iff \bar{L} \in C$$

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Closure Properties

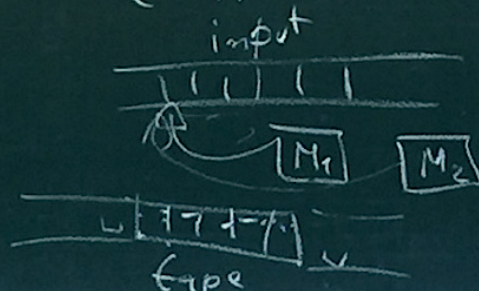
2. Union $C_1, C_2 \subseteq 2^{\Sigma^*}$

$$C_1 \vee C_2 := \{L_1 \cup L_2 \mid L_1 \in C_1, L_2 \in C_2\}$$

$$\chi_{L_1 \cup L_2}(x) = \begin{cases} 1, & x \in L_1 \text{ or } x \in L_2 \\ 0, & \text{else} \end{cases}$$

C is closed under Union if $C \vee C = C$

$$L(M_1) = L_1, L(M_2) = L_2$$



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3. Intersection

$$C_1 \cap C_2 := \{L_1 \cap L_2 \mid L_1 \in C_1, L_2 \in C_2\}$$

$$\chi_{L_1 \cap L_2} = \begin{cases} 1, & x \in L_1 \wedge x \in L_2 \\ 0, & \text{else} \end{cases}$$

closed under Intersection: $C \cap C = C$

De Morgan: $\text{co} - (\text{co} - C \vee \text{co} - C) = C \cap C$

$$\neg(\overline{L_1} \vee \overline{L_2}) = L_1 \wedge L_2$$

4. Concatenation

$$L_1 \circ L_2 := \{w_1 \circ w_2 \mid w_1 \in L_1, w_2 \in L_2\}$$

C is closed under concatenation if

$$\forall L_1, L_2 \in C \Rightarrow L_1 \circ L_2 \in C$$

regular expression

$$R = \begin{cases} \varepsilon \\ a, & a \in \Sigma \\ R_1 \vee R_2, & R_1, R_2 \\ R_1 \circ R_2 \\ R_1^* \end{cases}$$

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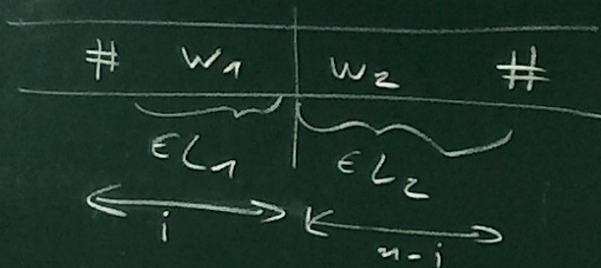
Closure Properties

closed under	REG	L	P	NP	PSPACE
Complement	✓	✓	✓	?	✓
Union	✓	✓	✓	✓!	✓
Intersection	✓	✓	✓	✓!	✓
Concatenation	✓	✓!	✓!	✓!	✓!

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1. L is closed under concatenation

$L_1, L_2 \in \mathcal{L} = \text{LOGSPACE}$
 M_1 and M_2 are logspace DTM



input: $w \in \Sigma^*$

1. For $i := 0$ to $|w| = n$ do
2. partition $w = w_1 \circ w_2$ with $w_1 \in \Sigma^i$
3. Run M_1 on w_1
4. Run M_2 on w_2
5. if both accept then accept
6. od
7. Reject

Counting takes $\log |w|$ space

if we store i binary: $|\text{bin}(|w|)| = \log |w|$

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2. NP is closed under Union

$L_1, L_2 \in NP$, $M_{1/2}$ NTMs for L_1, L_2
Pol. time bounded



$M(w)$

1. Guess $i \in \{1, 2\}$
2. Compute $M_i(x)$

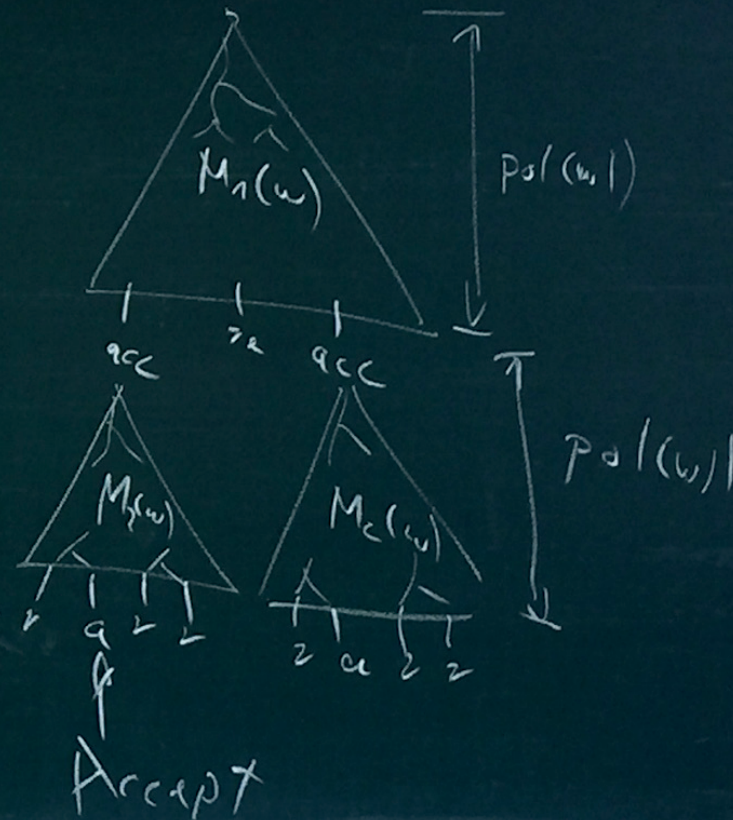
$M(w)$ accept if
 $(M_1(w) \text{ acc}) \vee (M_2(w) \text{ acc.})$

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3. NP is closed under intersection

M_1, M_2 are poly-time bound NTMs for L_1, L_2
input w .

1. Run $M_1(w)$
2. if $M_1(w)$ rejects then reject
3. Run $M_2(w)$
4. if $M_2(w)$ accepts then Accept
5. else reject



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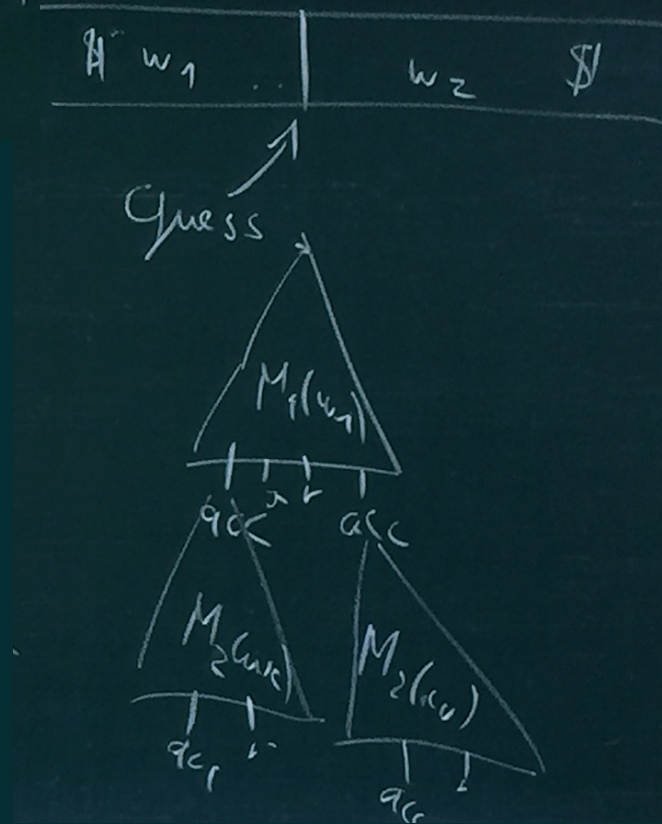
4. NP is closed under concatenation

$L_1, L_2 \in \mathcal{N}$, M_1, M_2 are pol-time bounded

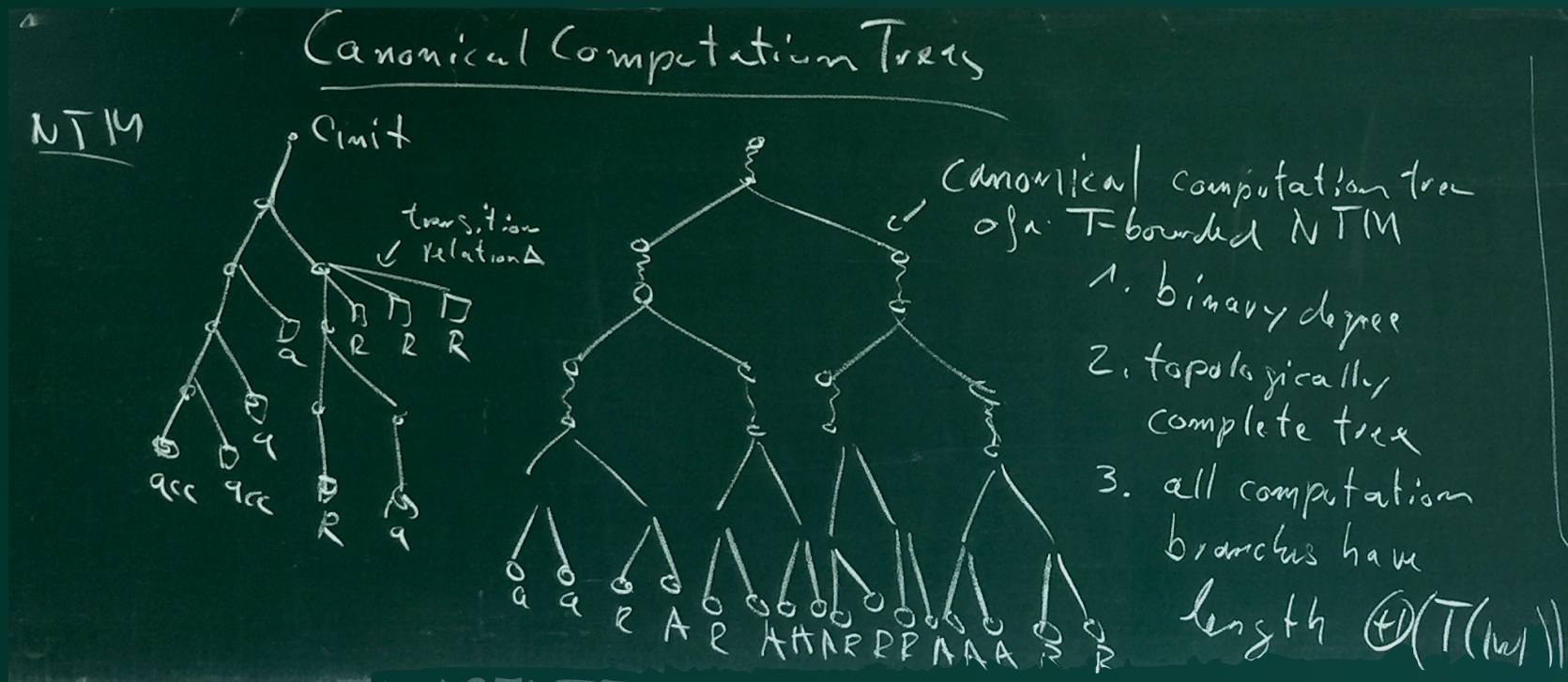
NTMs for L_1, L_2

input $w \in \Sigma^*$

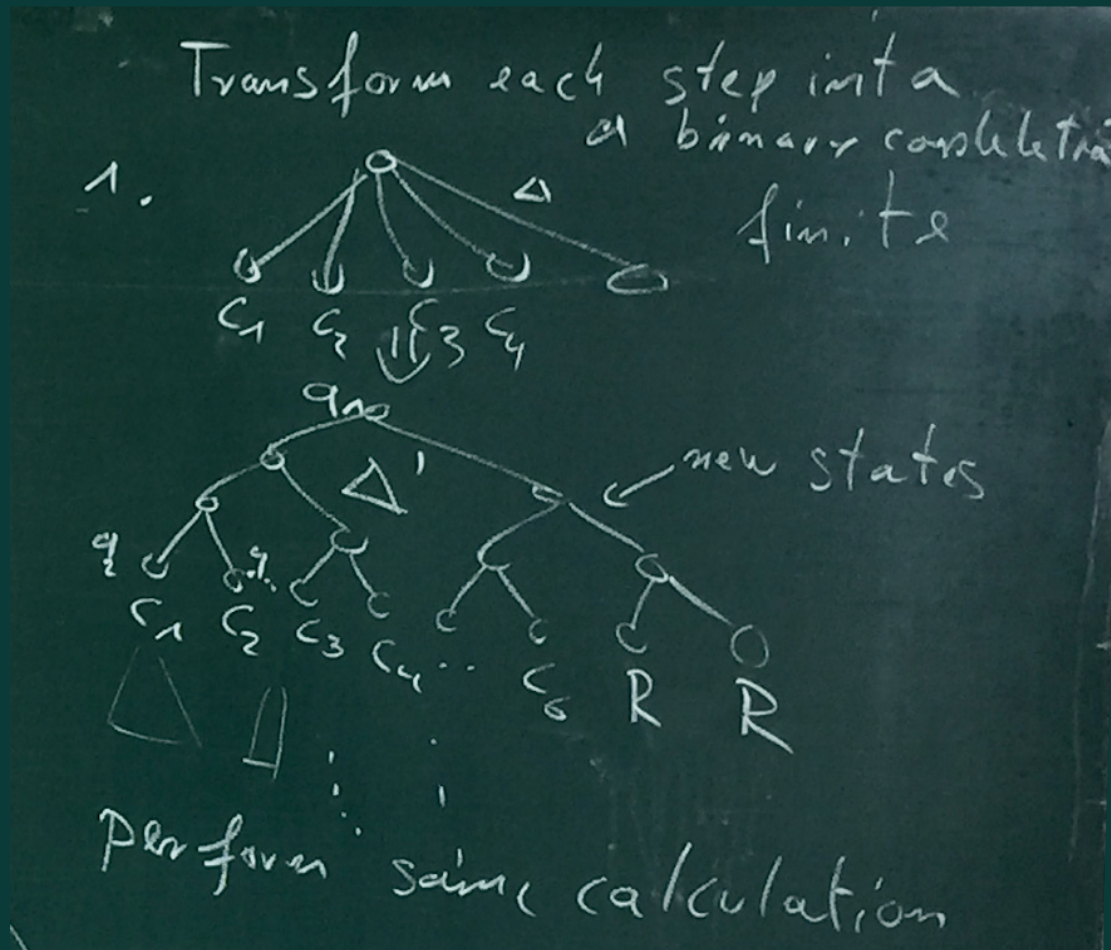
1. Guess $i \in \{0, \dots, |w|\}$
2. Partition $w_1 w_2 = w$, $w_1 = \epsilon^i$
3. Run $M_1(w_1)$
4. if $M_1(w_1)$ accepts then
5. Run $M_2(w_2)$
6. if $M_2(w_2)$ accepts then accept
7. fi
8. Reject



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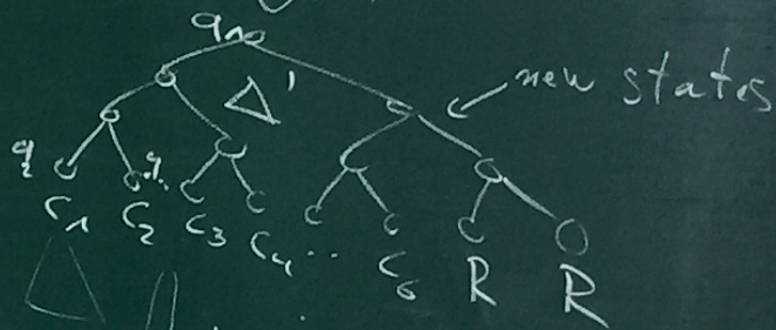
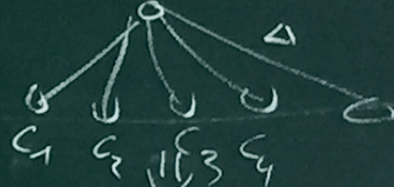
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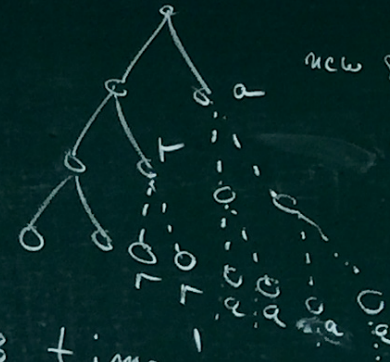
Transform each step into a
a binary computation
finite

1.



Perform same calculation

2. complete tree: introduce
take comp. tree



3 time

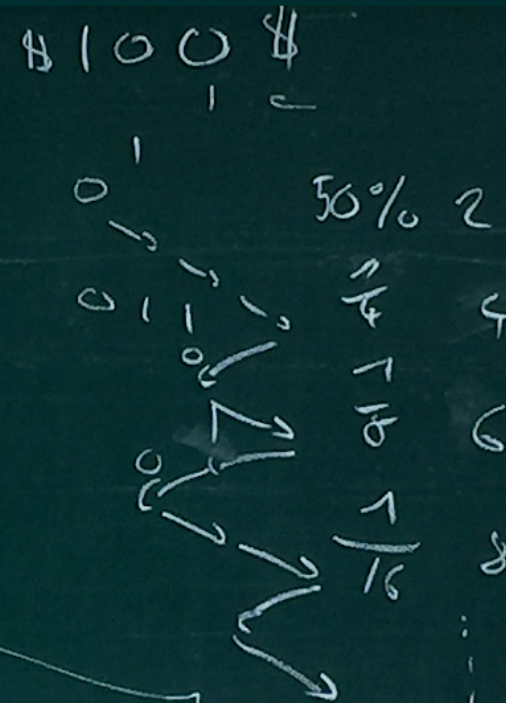
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3. Counter for time
- When to stop?

Counter tape for time
0 1 1 1 T(|w|)

$L \in \text{NTIME}(T)$

1. compute $T(|w|)$
2. Store it on the counter tape
3. count backwards to 0 in each step
4. if $M(w)$ accept continue
5. if count = 0 $\xrightarrow{\text{reject}}$ acc/nei



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T is time-constructable if $T(|w|)$ can be computed by a DTM in time $T(|w|)$.

Lemma T is time-constructable
 $L \in \text{NTIME}(T)$
then there exist a T -time bounded
NTM for L with a
canonical computation
tree.