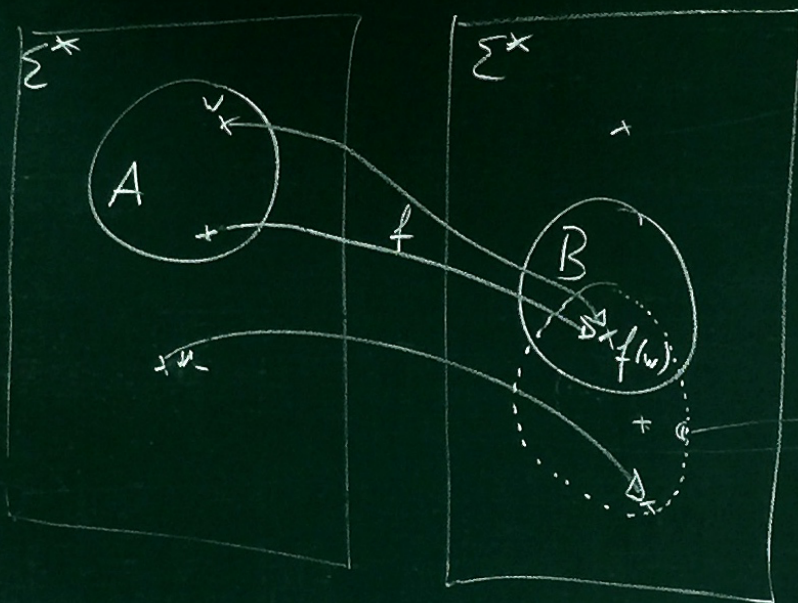


## Reductions



### 1. Many-one Reduction Mapping

$$A \leq_m B$$

$$A \subseteq \Sigma_1^*, B \subseteq \Sigma_2^*$$

$$\exists f: \Sigma_1^* \rightarrow \Sigma_2^*$$

$f$  can be computed by a TM. and

$$f^{-1}(B) = A$$

$$\forall x \in A : f(x) \in B$$

$$\forall x \notin A : f(x) \notin B$$

Theorem -  $B \in REC$  and  $A \leq_m B$   
 $\Rightarrow A \in REC$

-  $B \in RE$  and  $A \leq_m B$   
 $\Rightarrow A \in RE$



# 9 Reductions, OTMs and P<sup>NP</sup>

$$A \leq_m^P B$$

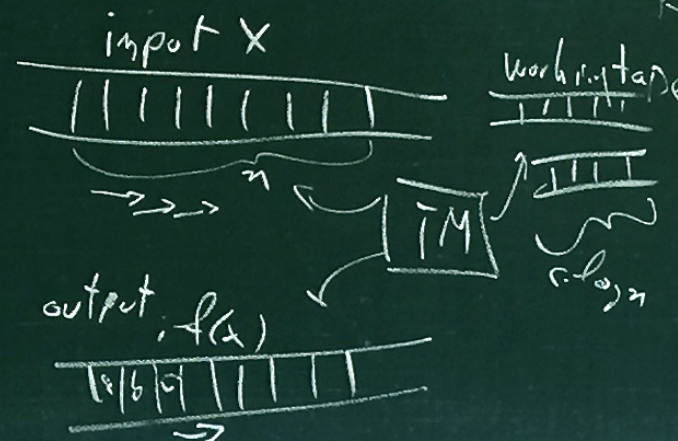
A can be poly time reduced to B

if  $\exists f: \Sigma_1^* \rightarrow \Sigma_2^*$

$f(x)$  can be computed by  
 a  $|x|^c$ -time bounded DTM  
 for  $c \in O(1)$ .

$$A \leq_m^{\log} B$$

if ... reduction which  
 is log space computable



# 9 Reductions, OTMs and P<sup>NP</sup>

Theorem  $B \in P$ ,  
 $A \leq_m^P B \Rightarrow A \in P$

$M$  is a DTM for  $B$  with time  
 $n^c$ .

$f$ : reduction and  $M_f$   
 computes  $f$  in time  $n^{c_f}$

$\forall A \in P, \exists B \in \text{TIME}(n)$   
 $A \leq_m^P B$

$M'$  input  $x$

1. compute  $f(x)$  using  $M_f$
2. compute  $M(f(x))$
3. copy output

correctness ✓

time:  $n^c + m^{c_f} \Rightarrow$

$$m = |f(x)| \leq n^c$$

$$x = n^c + n^{c \cdot c_f}$$



# 9 Reductions, OTMs and P<sup>NP</sup>

$$\underline{SAT \leq_m^P 3\text{-SAT}}$$

$SAT = \{ g \in \{0,1,v,\wedge,\neg,x,(,)\} \mid$   
 $g \text{ is satisfiable and}$   
 $\text{a Boolean function} \}$

$$x_0 \vee (x_1 \wedge \neg(x_{11} \vee x_{10})) \vee \neg(x_0 \wedge \neg x_1)$$

- context free grammar

$$G \rightarrow 0 \mid 1 \mid \text{var} \mid \text{lit} \mid$$

$$T_{AND} \mid T_{OR} \mid T_{neg}$$

$$\text{var} \rightarrow x \text{ number}$$

$$\text{number} \rightarrow 0 \mid 1 \mid \text{number}0 \mid$$

$$\text{number}1$$

$$\text{lit} \rightarrow \text{var} \mid \neg \text{var}$$

$$T_{and} \rightarrow (G) \wedge (G)$$

$$T_{OR} \rightarrow G \vee G$$

$$T_{neg} \rightarrow \neg(G)$$

# 9 Reductions, OTMs and P<sup>NP</sup>

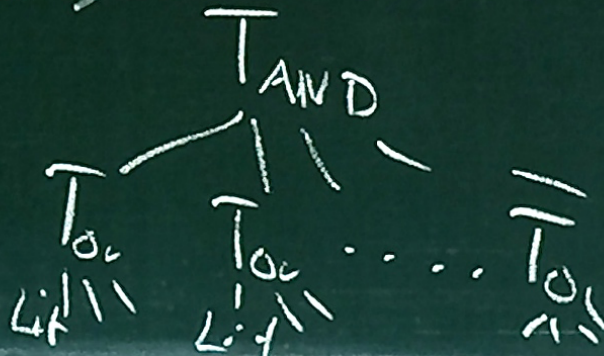
$g$  is satisfiable if

$$\exists x_1, x_2, \dots, x_m \in \{0, 1\} : g(x_1, \dots, x_m) = 1$$

3-SAT  $\in$  CNF

$g'(x_1, \dots, x_n) =$  conjunctive normal form

$$(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \bar{x}_5 \vee \bar{x}_6)$$





$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$\neg (a \vee b) = \neg a \wedge \neg b$$

$$(x_1 \vee x_2 \vee x_3 \vee x_4)$$

Gadget  $\rightarrow$  helping variables  $h_1, h_2$

# 9 Reductions, OTMs and P<sup>NP</sup>

$\exists x_1, x_2: h = \overline{x_1 \vee x_2}$

$\Leftrightarrow h = \overline{T}$

	$\overline{T}$	$T$
$\overline{h}$	0	1
$h$	1	0

$(\overline{h} \vee \overline{T}) \wedge (h \vee T)$

$(\overline{h} \vee x_1 \vee x_2) \wedge (h \vee \overline{x_1} \vee \overline{x_2})$

$\Leftrightarrow (\overline{h} \vee x_1 \vee x_2) \wedge (h \vee \overline{x_1} \wedge \overline{x_2})$

$\Leftrightarrow (\overline{h} \vee x_1 \vee x_2) \wedge (h \vee \overline{x_1}) \wedge (h \vee \overline{x_2})$

$(\overline{h} \vee x_1 \vee x_2) \wedge (h \vee \overline{x_1} \vee h) \wedge (h \vee \overline{x_2})$

$h = x_1 \wedge x_2$

$\Leftrightarrow (\overline{h} \vee (x_1 \wedge x_2)) \wedge (h \vee \overline{x_1} \wedge \overline{x_2})$

$\Leftrightarrow (\overline{h} \vee x_1) \wedge (\overline{h} \vee x_2) \wedge (h \vee \overline{x_1} \vee \overline{x_2})$

$\Leftrightarrow (\overline{h} \vee \overline{h} \vee x_1) \wedge (\overline{h} \vee \overline{h} \vee x_2) \vee (h \vee \overline{x_1} \vee \overline{x_2})$

---

$h = \overline{x_1} \vee x_2, h = \overline{x_1} \vee \overline{x_2}, h = \overline{x_1}$



# 9 Reductions, OTMs and P<sup>NP</sup>

$SAT \leq_m^P 3-SAT$

SAT-formula

3-SAT  $\leq_m^P$  SAT

Red. fkt.

$f(x) = x$

---

$SAT \equiv_m^P 3-SAT$

gadget

$h_2 = h_5 \vee h_6$

$h_6 = x_1 \wedge \bar{x}_2$

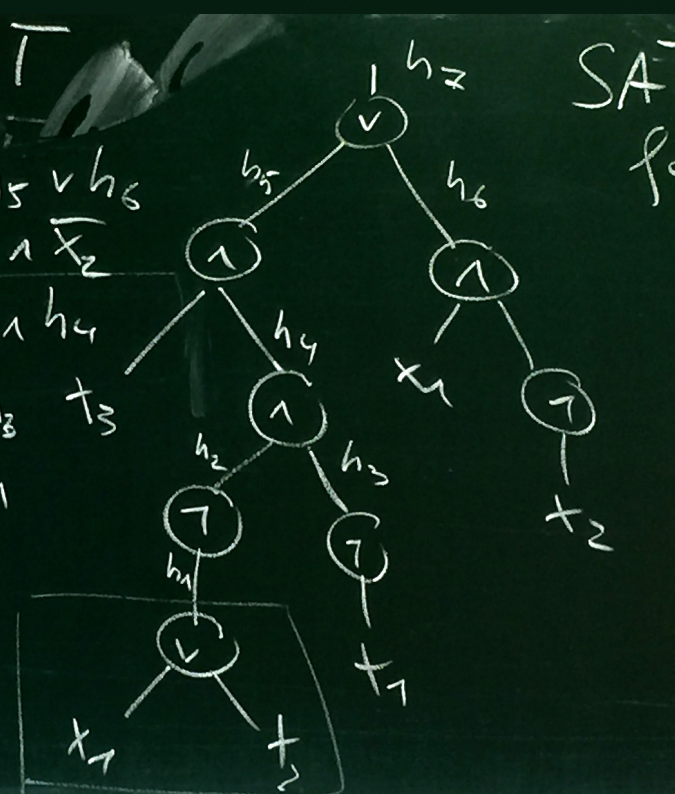
$h_5 = x_3 \wedge h_4$

$h_4 = h_2 \wedge h_3$

$h_3 = \neg x_1$

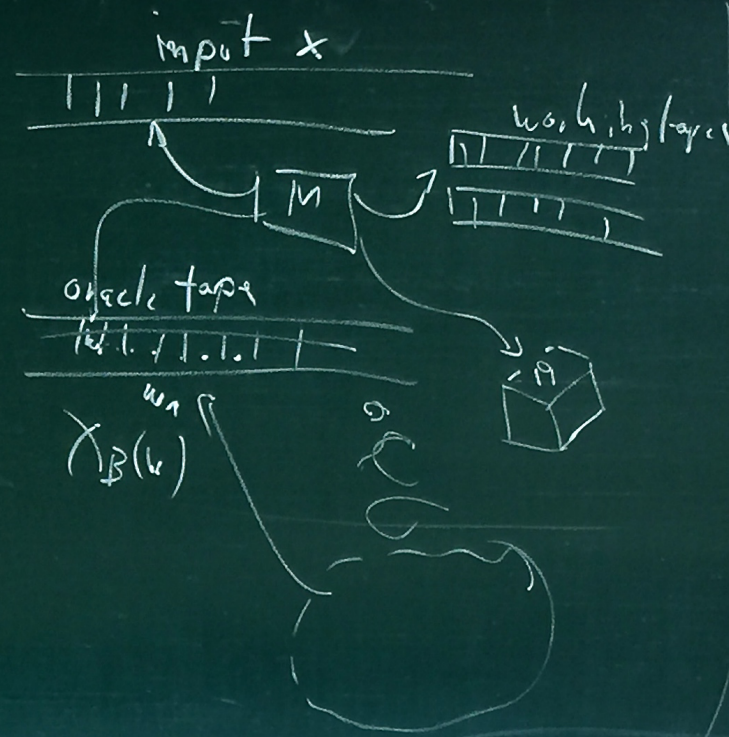
$h_2 = \neg h_1$

$h_1 = x_1 \vee x_2$



## Def OTM

An oracle TM with oracle  $B \in \Sigma^*$ 
  
 has a special state  $q_{\text{oracle}}$ ; if it
   
 goes into this step (only once)
   
 then one tape contents  $w$  is replaced
   
 by  $\chi_B(w)$ .





input  $x$

1. Do something

2. Ask the oracle

3. Do something with it

4. output result

Def Turing reduction  $A \leq_T B$

if  $\exists$  OTM  $M$  with oracle  $B$   
that decides  $A$ .

Theorem  $B \in P, A \leq_T^P B$   
 $\Rightarrow A \in P$

$C_1 \subset C_2 := \{L \subseteq \Sigma^* \mid$

$\exists \text{ TM } M \text{ with oracle } A$   
 $L \in C_2 : L(M) = L\}$

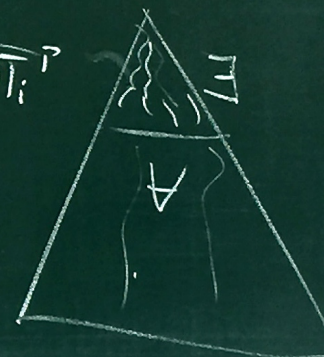
$$P^P = P$$

$$\Delta_2^P = P^{NP}$$

$$\Sigma_2^P = NP^{NP}$$

$$\cong NP \cup co-NP$$

$$\Delta_{i+1}^P = P^{\Sigma_i^P} = P^{\Pi_i^P}$$





# 9 Reductions, OTMs and $P^{NP}$

