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Distributed Storage Networks and Computer Forensics

8 Analysis of DHT

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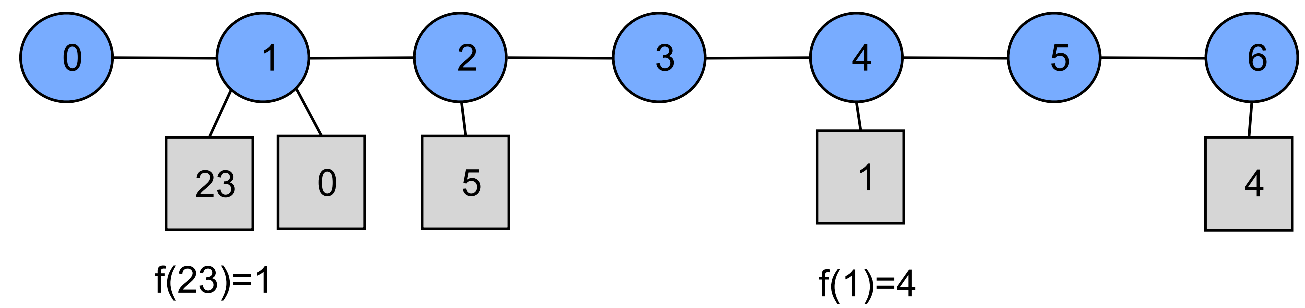
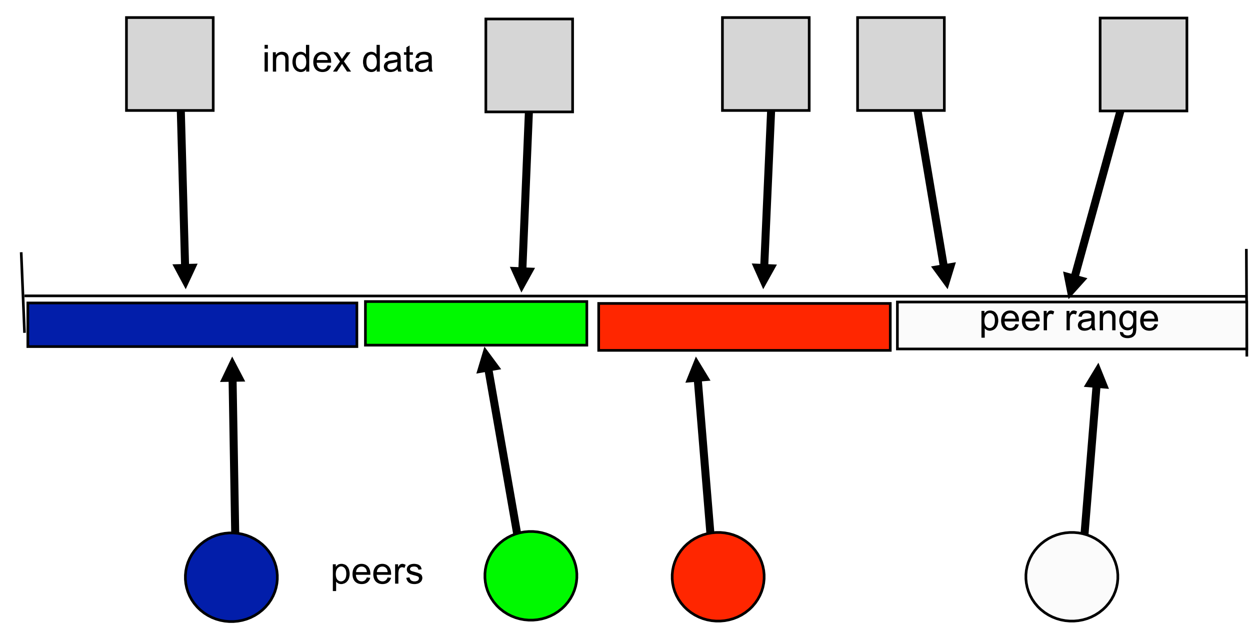
Distributed Hash-Table (DHT)

- **Hash table**
  - does not work efficiently for inserting and deleting
- **Distributed Hash-Table**
  - servers are „hashed“ to a position in an continuos set (e.g. line)
  - data is also „hashed“ to this set
- **Mapping of data to servers**
  - servers are given their own areas depending on the position of the direct neighbors
  - all data in this area is mapped to the corresponding server
- **Literature**
  - *“Consistent Hashing and Random Trees: Distributed Caching Protocols for Relieving Hot Spots on the World Wide Web”*, David Karger, Eric Lehman, Tom Leighton, Mathhew Levine, Daniel Lewin, Rina Panigrahy, STOC 1997

Pure (Poor) Hashing

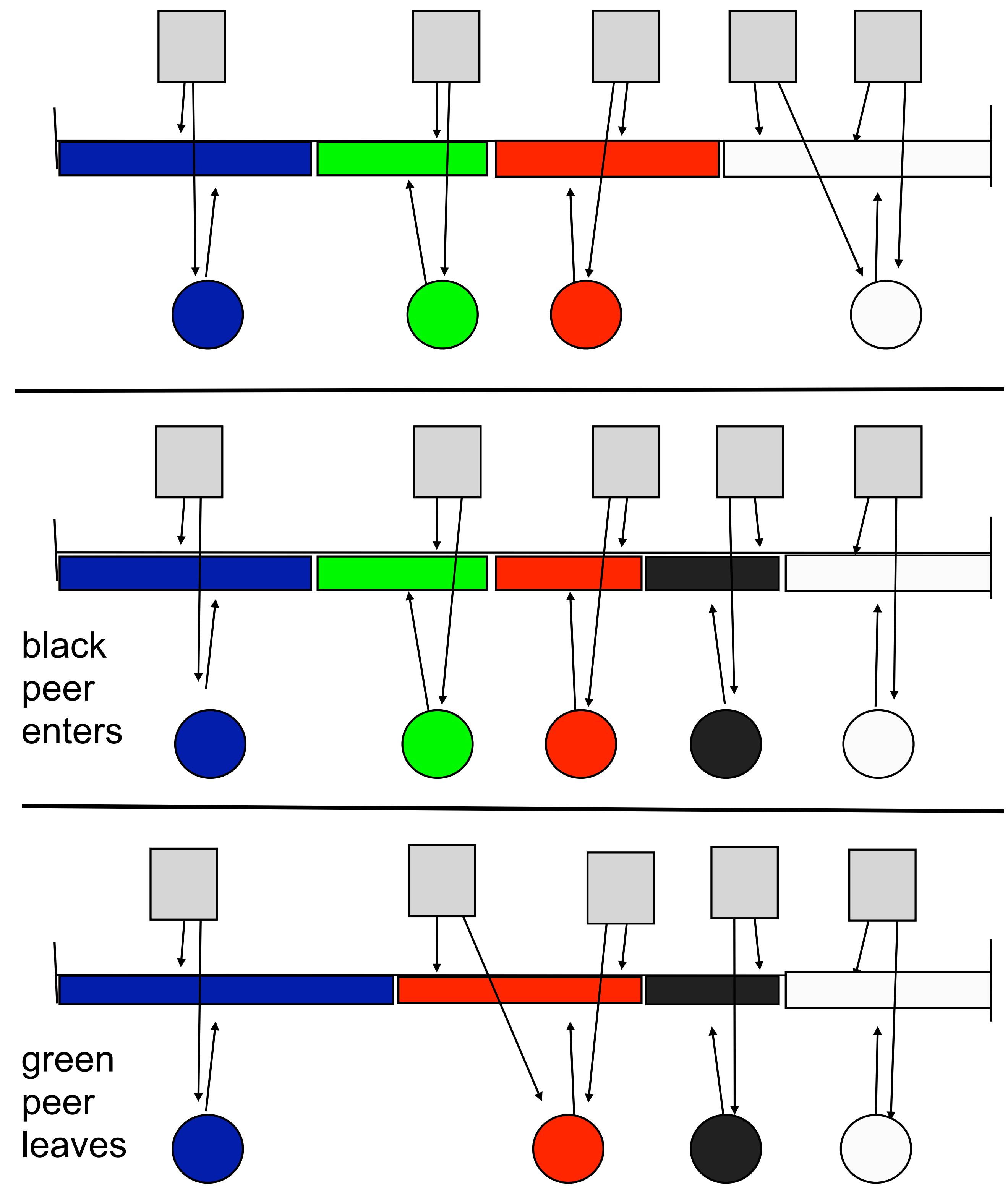


DHT

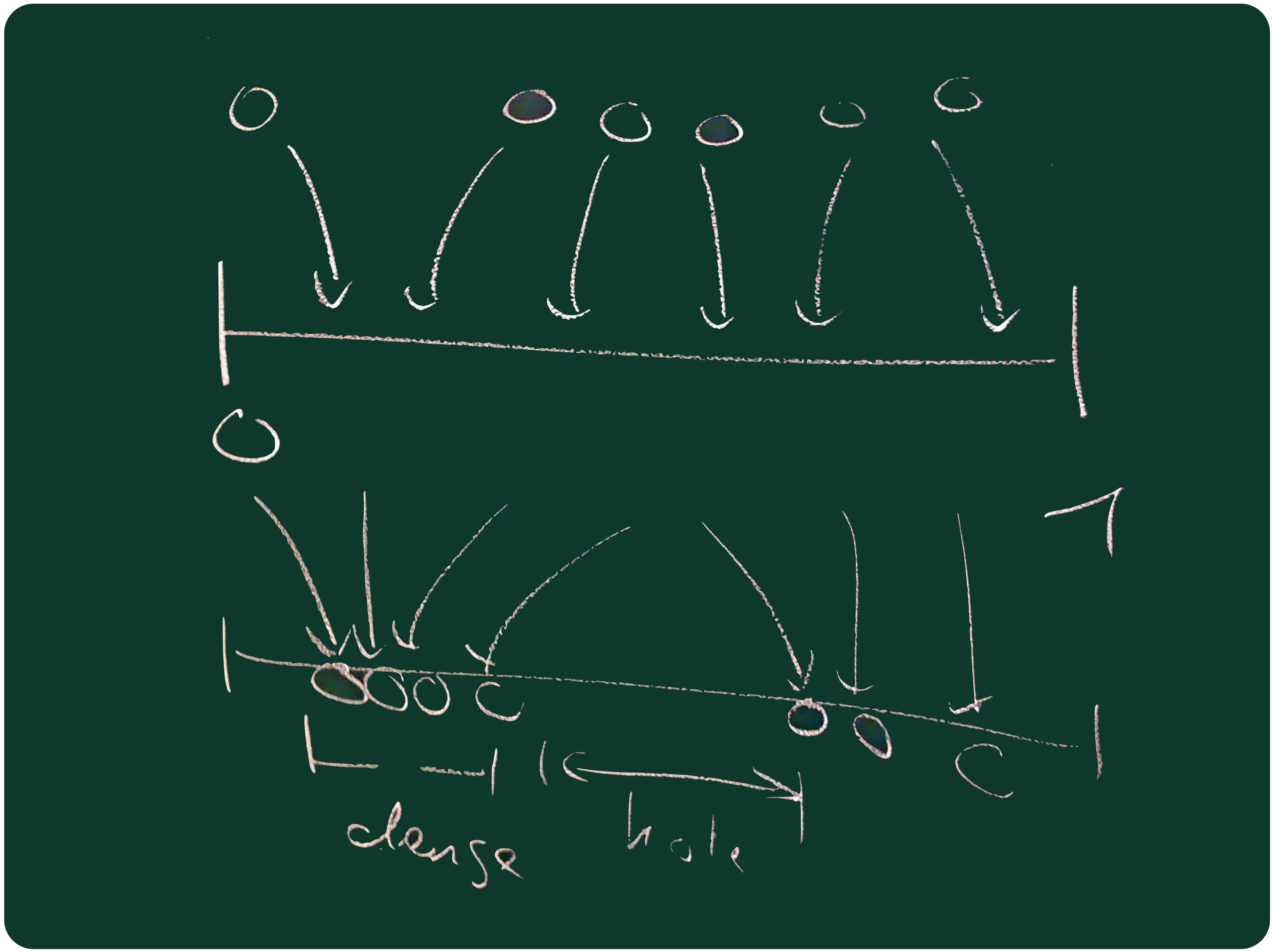


Entering and Leaving a DHT

- **Distributed Hash Table**
  - devices are hashed to to position
  - blocks are hashed according to the ID
- **When a device is added**
  - only blocks from neighbors have to be moved
- **When a device is deleted**
  - blocks are moved only to the neighbors



Holes and Dense Areas



Size of Holes

- **Theorem**
  - If n elements are randomly inserted into an array [0,1[then with constant probability there is a „hole“ of size Ω(log n/n), i.e. an interval without elements.
- **Proof**
  - Consider an interval of size log n / (4n)
  - The chance not to hit such an interval is (1-log n/(4n))
  - The chance that n elements do not hit this interval is

$$\left(1-\frac{\log n}{4n}\right)^{n} = \left(1-\frac{\log n}{4n}\right)^{\frac{4n}{\log n}\,\frac{\log n}{4}} \geq \left(\frac{1}{4}\right)^{\frac{1}{4}\log n} = \frac{1}{\sqrt{n}}$$

  - The expected number of such intervals is more than 1.
  - Hence the probability for such an interval is at least constant.

Proof of Dense Areas

$$\left(\frac{1}{4}\right)^{\frac{1}{4}\cdot \log n} = 2^{\left(\frac{1}{4}\log n\right)\overbrace{\log_2 \frac{1}{4}}^{-2}}$$

$$= 2^{\left(-\frac{1}{2}\right)\cdot \log n}$$

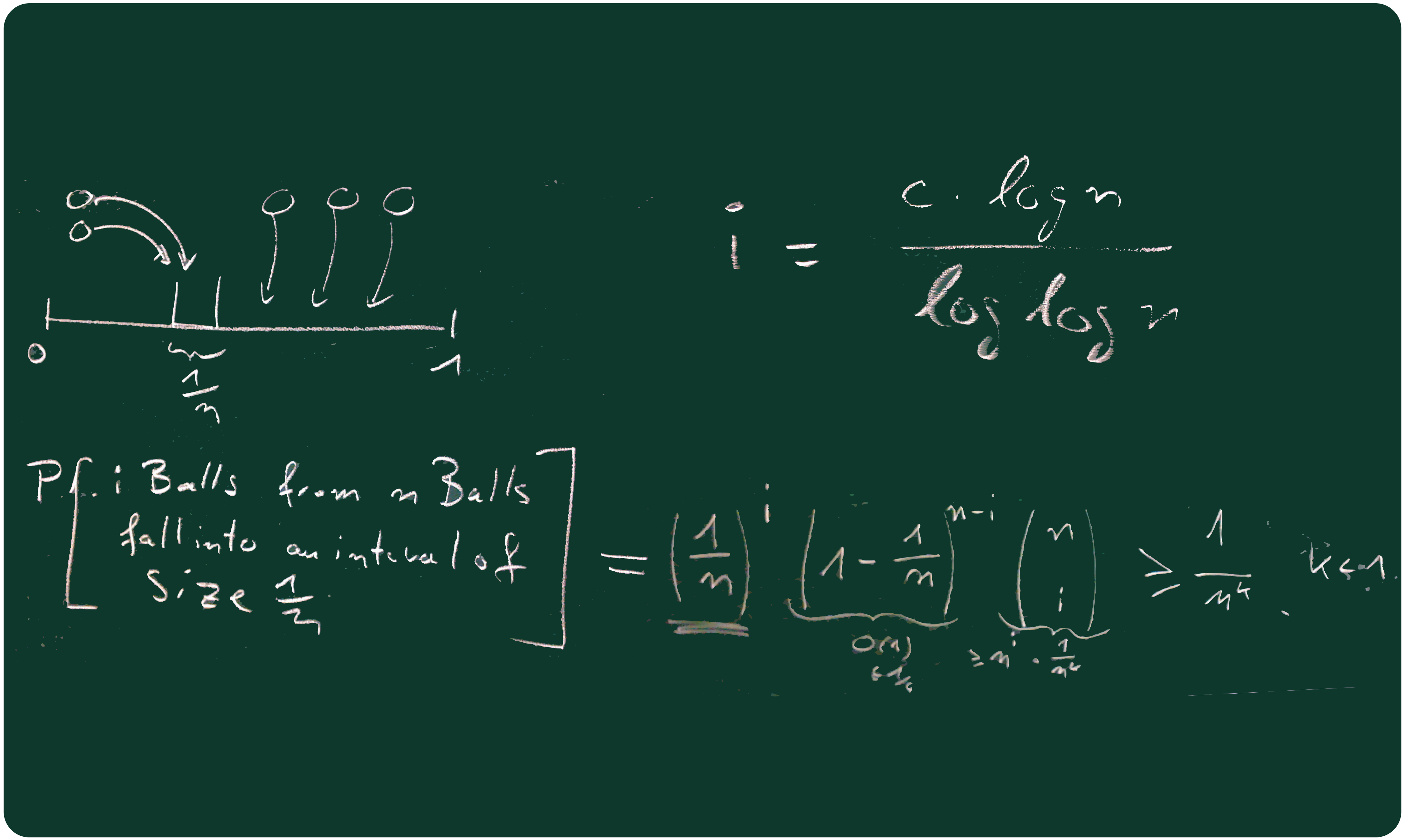
$$= n^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}$$

Expectation: $\frac{4n}{\log n}\cdot\frac{1}{\sqrt{n}} = \frac{4\sqrt{n}}{\log n}$

Dense Spots

- **Theorem**
  - If n elements are randomly inserted into an array [0,1[then with constant probability there is a dense interval of length 1/n with at least Ω(log n/ (log log n)) elements.
- **Proof**
  - The probability to place exactly i elements in to such an interval is $$\left(\frac{1}{n}\right)^i \left(1-\frac{1}{n}\right)^{n-i} \binom{n}{i}$$
  - for i = c log n / (log log n) this probability is at least $1/n^k$ for an appropriately chosen c and k<1
  - Then the expected number of intervals is at least 1

Proof of Dense Areas



Proof of Dense Areas

$$\frac{1}{4} \underset{m \geq 2}{\leq} \left(1 - \frac{1}{m}\right)^m \leq \frac{1}{e}$$

$$\left(1 - \frac{1}{n}\right)^{n-i} = \left(1 - \frac{1}{n}\right)^{n \frac{n-i}{n}}$$

$$\geq \left(\frac{1}{4}\right)^{1 - \frac{i}{n}}$$

$$\geq \frac{1}{4}$$

Proof of Dense Areas

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-i+1)}{i!}$$

$$\geq \frac{\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-i+1}{n}}{i^i}\; n^i \qquad \frac{i}{n} \leq \frac{1}{2}$$

$$\geq \underbrace{\left(1-\frac{i-1}{n}\right)^{n-i}} \cdot \frac{n^i}{i^i}$$

$$\left(1-\frac{i-1}{n}\right)^{\frac{n}{i-1}\cdot\frac{n-i}{n}(i-1)} \geq \left(\frac{1}{4}\right)^{\left(1-\frac{i}{n}\right)(i-1)} \geq \left(\frac{1}{4}\right)^{\frac{1}{2}i} = \left(\frac{1}{2}\right)^{i}$$

Proof of Dense Areas

$$\left(\frac{1}{2}\right)^i \cdot \frac{1}{i^i} = 2^{-i - i\cdot\ln i} \geq 2^{\underbrace{-k\cdot\log n}_{\frac{1}{n^k}}}$$

$$i + i\cdot\ln i \leq \frac{c\cdot\log n}{\log\log n}\left(1+\ln c+\ln\log n - \ln\log\log n\right)$$

$$i(1+\ln i)$$

$$\leq \frac{c\cdot\log n}{\log\log n}\left(1+\ln c+(\ln 2)\right)\log\log n$$

$$\hat{=}\ \overbrace{c(1+\ln c+\ln 2)}^{k}\cdot\log n$$

Averaging Effect

- **Theorem**
  - If Θ(n log n) elements are randomly inserted into an array [0,1[then with high probability in every interval of length 1/n there are Θ(log n) elements.

Excursion

- **Markov-Inequality**
  - For random variable X>0 with E[X] > 0:

$$\mathbf{P}[X \geq k \cdot \mathbf{E}[X]] \leq \frac{1}{k}$$

- **Chebyshev**

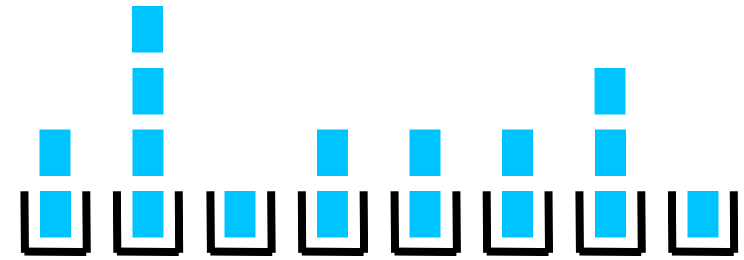
$$\mathbf{P}[|X - \mathbf{E}[X]| \geq k] \leq \frac{\mathbf{V}[X]}{k^2}$$

  - for Variance $\mathbf{V}[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$
- **Stronger bound: Chernoff**

Chernoff-Bound

- **Theorem Chernoff Bound**
  - Let $x_1,\dots,x_n$ independent Bernoulli experiments with
    - $P[x_i = 1] = p$
    - $P[x_i = 0] = 1-p$
  - Let $$S_n = \sum_{i=1}^{n} x_i$$
  - Then for all c>0
$$\mathbf{P}[S_n \geq (1+c) \cdot \mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{3}\min\{c,c^2\}pn}$$
  - For 0≤c≤1
$$\mathbf{P}[S_n \leq (1-c) \cdot \mathbf{E}[S_n]] \quad \leq \quad e^{-\frac{1}{2}c^2 pn}$$

Balls and Bins



Lemma

If m= k n ln n Balls are randomly placed in n bins:

1. Then for all c>k the probability that more than c ln n balls are in a bin is at most O($n^{-c'}$) for a constant c'>0.
2. Then for all c<k the probability that less than c ln n balls are in a bin is at most O($n^{-c'}$) for a constant c'>0.

Proof:

Consider a bin and the Bernoulli experiment B(k n ln n,1/n) and expectation: μ = m/n = k ln n

1. Case: c>2k
$$\mathrm{P}[X \geq c \ln n] = \mathrm{P}[X \geq (1+(c/k-1))k \ln n]$$
$$\leq e^{-\frac{1}{3}(c/k-1)k \ln n} \leq n^{-\frac{1}{3}(c-k)}$$

2. Case: k<c<2k
$$\mathrm{P}[X \geq c \ln n] = \mathrm{P}[X \geq (1+(c/k-1))k \ln n]$$
$$\leq e^{-\frac{1}{3}(c/k-1)^2 k \ln n} \leq n^{-\frac{1}{3}(c-k)^2,}$$

3. Case: c<k
$$\mathrm{P}[X \leq c \ln n] = \mathrm{P}[X \leq (1-(1-c/k))k \ln n]$$
$$\leq e^{-\frac{1}{2}(1-c/k)^2 k \ln n} \leq n^{-\frac{1}{2}(k-c)^2/k}$$

Concept of High Probability

Lemma

If A(i) holds with **high** probability, i.e. $1-n^{-c}$, then

(A(1) and A(2) and ... and A(n)) with **high** probability,
i.e. $1-n^{-(c-1)}$

Proof:

- For all i: $P[\neg A(i)] \leq n^{-c}$
- Hence: $P[\neg A(1) \text{ or } \neg A(2) \text{ or } \ldots \neg A(n)] \leq n \cdot n^{-c}$

$$P[\neg(\neg A(1) \text{ or } \neg A(2) \text{ or } \ldots \neg A(n))] \leq 1 - n \cdot n^{-c}$$

DeMorgan:

$$P[A(1) \text{ and } A(2) \text{ and } \ldots A(n)] \leq 1 - n \cdot n^{-c}$$

Principle of Multiple Choice

- **Before inserted check c log n positions**
- **For position p(j) check the distance a(j) between potential left and right neighbor**
- **Insert element at position p(j) in the middle between left and right neighbor, where a(j) was the maximum choice**
- **Lemma**
  - After inserting n elements with high probability only intervals of size 1/(2n), 1/n und 2/n occur.

Proof of Lemma

1. Part: With high probability there is no interval of size larger than 2/n

follows from this Lemma

Lemma*

Let c/n be the largest interval. After inserting 2n/c peers all intervals are smaller than c/(2n) with high probability

From applying this lemma for c=n/2,n/4, ...,4 the first lemma follows.

Proof

- **2nd part: No intervals smaller than 1/(2n) occur**
  - The overall length of intervals of size 1/(2n) before inserting is at most 1/2
  - Such an area is hit with probability at most 1/2
  - The probability to hit this area more than c log n times is at least

$$2^{-c \log n} = n^{-c}$$

  - Then for c>1 such an interval will not further be divided with probability into an interval of size 1/(4m).

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Algorithms and Methods for Distributed Storage Networks

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