

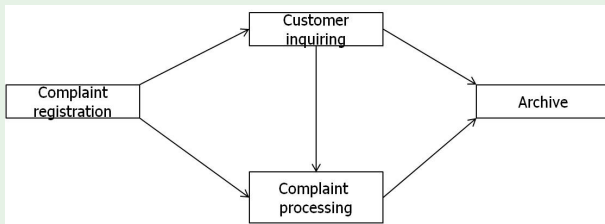
Chapter 12: Modeling and Analysis of Distributed Applications

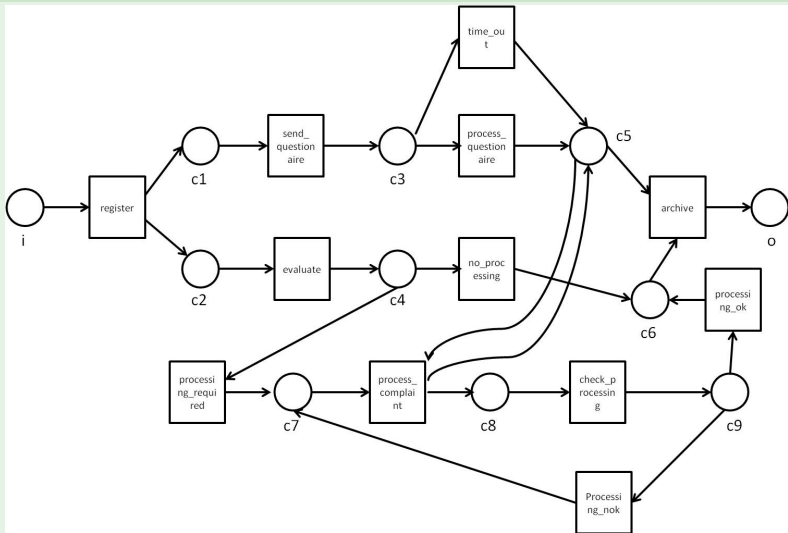
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Petri-Nets

- Petri-nets are abstract formal models capturing the flow of information and objects in a way which makes it possible to describe distributed systems and processes at different levels of abstraction in a unified language.
- Petri-nets have the name from their inventor Carl Adam Petri, who introduced this formalism in his PhD-thesis 1962.

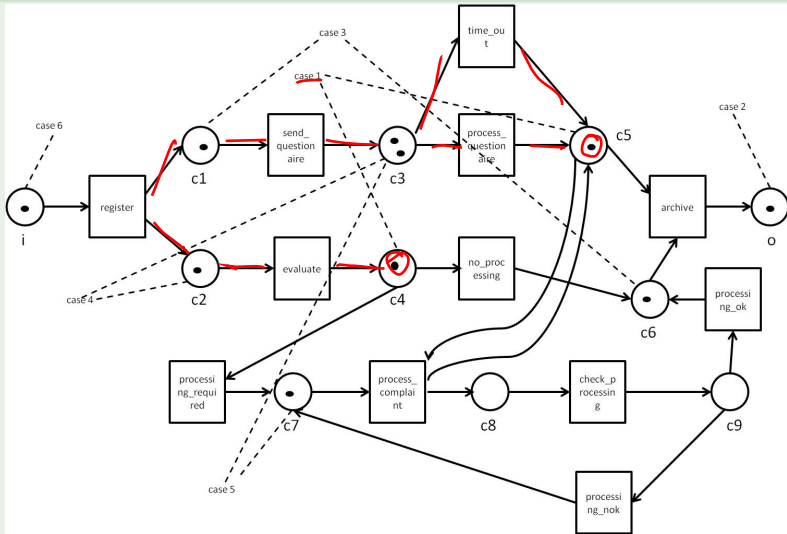
Processing of complaints: informal description.



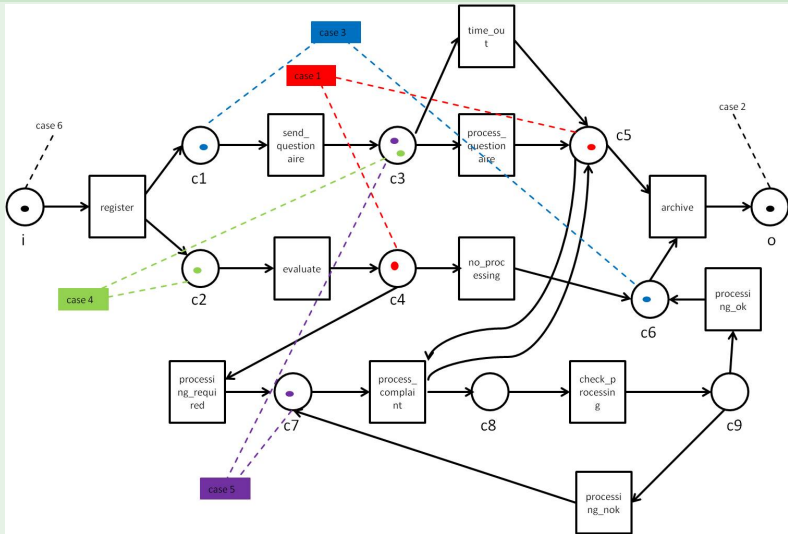
Complaints processing: formal Petri-net orchestration.¹

¹van der Aalst: The Application of Petri nets to Workflow Management. Journal of Circuits, Systems, and Computers 8(1): 21-66 (1998)

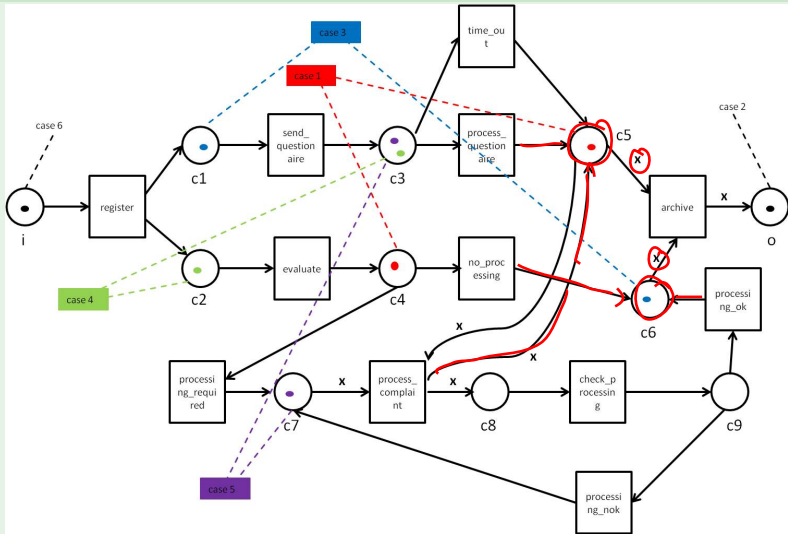
Complaints processing: more than one complaint



Complaints processing: how to distinguish complaints



Complaints processing: keeping things together



Petri-nets

Petri-nets model system dynamics.

- Activities trigger state transitions,
- activities impose control structures,
- applicable for modelling discrete systems.

Benefits

- Uniform language,
- can be used to model sequential, causal independent (concurrent, parallel, nondeterministic) and monitored exclusive activities.
- open for formal analysis, verification and simulation,
- graphical intuitive representation.

The name *Petri-net* denotes a variety of different versions of nets - we will discuss the special case of *System Nets* following the naming introduced by W. Reisig.

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Section 12.1 Elementary System Nets

UML \rightarrow sequence \leftrightarrow
 \searrow Activity


automata / state
m.

Basic elements of an elementary System Net (eS-Net)

- System states are represented by *places*, graphically circles or ovals. 
- A place may be marked by an arbitrary number of *tokens* graphically represented by black dots.
- System dynamics is represented by *transitions*, graphically rectangles.
- *Transitions* represent activities (events) and the causalities between such activities (events) are represented by edges.
- *Multiplicities* represent the consumption, respectively creation of resources which are caused by the *occurrence* of activities.

Section 12.1 Elementary System Nets

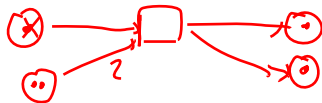
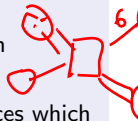
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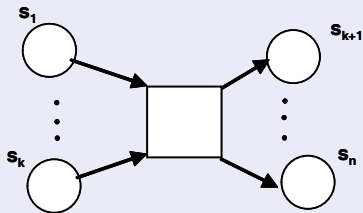
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A transition *may* occur when certain conditions with respect to the markings of its directly connected places are fulfilled; the *occurrence* of a transition - also called its *firing* - effects the markings of its directly connected edges, i.e. has local effects.

The *surrounding* of a transition t is given by t and all its directly connected places:

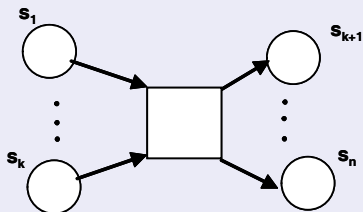


s_1, \dots, s_k are called *preconditions* (*pre-places*), s_{k+1}, \dots, s_n *postconditions* (*post-places*).

A place which is pre- and post-place at the same time is called a *loop*.

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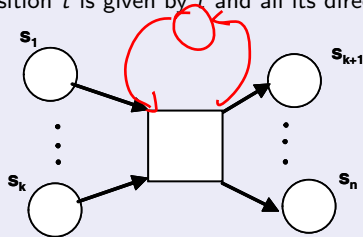


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A *net* is given as a triple $N = (\underline{P}, \underline{T}, \underline{F})$, where

- P , the set of *places*, and T , the set of *transitionen*, are non-empty disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$, is the set of directed edges, called *flow relation*, which is a binary relation such that $dom(F) \cup cod(F) = P \cup T$.
~ range

Let $N = (P, T, F)$ be a net and $x \in P \cup T$.

$$xF := \{y \mid (x, y) \in F\}$$

$$Fx := \{y \mid (y, x) \in F\}$$

For $p \in P$, pF is the set of *post-transitions* of p ; Fp is the set of *pre-transitions* of p .
 For $t \in T$, tF is the set of *post-places* of t ; Ft is the set of *pre-places* of t .

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Let $N = (P, T, F)$ be a net. Any mapping m from P into the set of natural numbers NAT is called a *marking* of P .

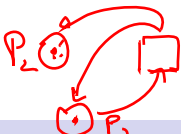
A mapping $P \rightarrow NAT \cup \{\omega\}$ is called ω -*marking*. ω represents an infinitely large number of tokens.

Arithmetic of ω :

$$\omega - n = \omega, \omega + n = \omega, n \cdot \omega = \omega, 0 \cdot \omega = 0, \omega > n$$

where $n \in NAT, n > 0$.

A *marking* represents a possible system state.



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- (P, T, F) a net,
- $V: F \rightarrow \text{NAT}^+$ a multiplicity,
- m_0 a marking called *initial marking*.

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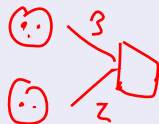
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A transition may fire once it is enabled.

Let $N = (P, T, F, V, m_0)$ a eS-Net, m a marking and $t \in T$ a transition.

- t is enabled at m , if for all pre-places $p \in Ft$ there holds:

$$m(p) \geq V(p, t).$$



- Whenever t is enabled at m , then t may fire at m . Firing t at m transforms m to m' , $m[t \rangle m'$, in the following way:

$$m'(p) := \begin{cases} m(p) - V(p, t) + V(t, p) & \text{if } p \in Ft, p \in tF, \\ m(p) - V(p, t) & \text{if } p \in Ft, p \notin tF, \\ m(p) + V(t, p) & \text{if } p \notin Ft, p \in tF, \\ m(p) & \text{otherwise.} \end{cases}$$

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Transitions and markings in terms of vectors

Let places in P be linearly ordered.

- Markings of a net can be considered as vectors of nonnegative integers of dimension $|P|$, called *place-vectors*.
- Transitions t can be characterized as vectors of nonnegative integers of dimension $|P|$, called *transition vectors* $\Delta t, t^+, t^-$:

Let $N = (P, T, F, V, m_0)$ a eS-Net, $p \in P$ and $t \in T$.

$$t^+(p) := \begin{cases} V(t, p) & \text{if } p \in tF, \\ 0 & \text{otherwise.} \end{cases}$$

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Place and transition vectors at work:

- $m \leq m'$, if $m(p) \leq m'(p)$ for $\forall p \in P$,
- $m < m'$, if $m \leq m'$, however $m \neq m'$.
- t is enabled at m iff $t^- \leq m$,
- $m[t \succ m'$ iff $t^- \leq m$ and $m' = m + \Delta t$.

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Reachability

Let $N = (S, T, F, V, m_0)$ a eS-Net.

We denote $W(T)$ the set of words with finite length over T ; $\epsilon \in W(T)$ is called the *empty word*.

The length of a word $w \in W(T)$ is given by $l(w)$. We have $l(\epsilon) = 0$.

Let m, m' be markings of P and $w \in W(T)$. We define a relation $m[w \succ m'$ inductively:

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The *reachability relation* $[* \succ$ of N is defined by

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- $L_N(m) := \{w \mid \exists m' : m[w \succ m']\}$, the set of all words representing firing sequences of transitions of N starting at m ,
- $\Delta w := \sum_{i=1}^n \Delta t_i$, wobei $w = t_1 t_2 \dots t_n$.

Results

- $[* \succ$ is reflexiv and transitiv.
- $m[w \succ m'] \Rightarrow (m + m^*)[w \succ (m' + m^*)], \forall m^* \in NAT^{|S|}$. (Monotonie)
- $m[w \succ m'] \Rightarrow m' = m + \Delta w$.

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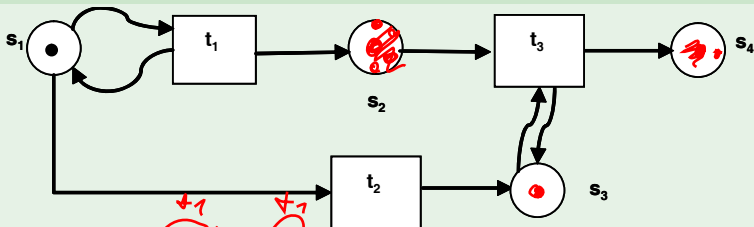
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Reachability graph

Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Reachability graph* of N is a directed graph $EG(N) := (R_N(m_0), B_N)$; $R_N(m_0)$ is the set of nodes and B_N is the set of annotated edges as follows:

$$B_N = \{((m) \downarrow t, (m')) \mid m, m' \in R_N(m_0), t \in T, m[t \succ m']\}.$$

Exercise: Give the reachability graph of the following eS-Net:



$$R_N(m_0) = \{ (1, 0, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0), (1, 3, 0, 0), \dots, \\ \underline{(0, 0, 1, 0)}, (0, 1, 1, 0), (0, 2, 1, 0), (0, 3, 1, 0), \dots, \\ \underline{(0, 0, 1, 1)}, (0, 1, 1, 1), (0, 0, 1, 2), (0, 2, 1, 1), (0, 1, 1, 2), (0, 0, 1, 3), \dots \}$$

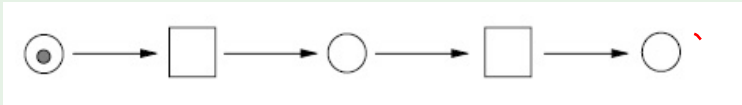
$$L_N(m_0) = \{ \epsilon, \underline{t_1}, t_1 t_1, t_1 t_1 t_1, \dots, \\ t_2, t_1 t_2, t_1 t_1 t_2, t_1 t_1 t_1 t_2, \dots, \\ t_1 t_2 t_3, t_1 t_1 t_2 t_3, t_1 t_1 t_2 t_3 t_3, t_1 t_1 t_1 t_2 t_3, t_1 t_1 t_1 t_2 t_3 t_3, \dots \}$$

Section 12.2 Control Patterns

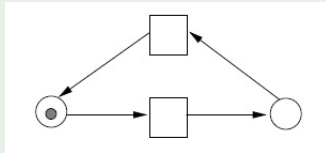
- eS-nets can be used to model *causal dependencies*; for modelling temporal aspects extensions of the formalism are required.
- Whenever between some transitions there are no causal dependencies, the transitions are called *concurrent*; concurrency is a prerequisite for parallelism.

Some typical causalities

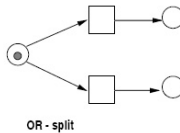
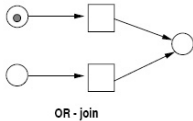
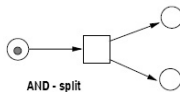
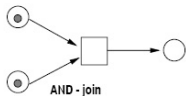
Sequence



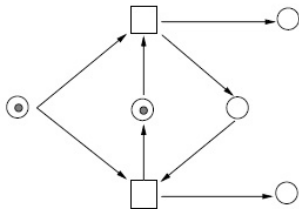
Iteration



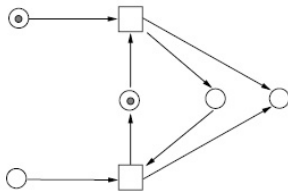
AND-join, OR-join, AND-split, OR-split



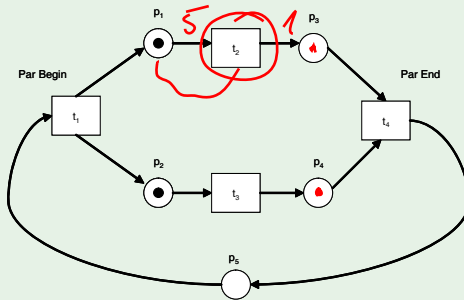
OR-Split with regulation



OR-Join with regulation



A eS-Net with concurrency



Section 12.3 Analysis

Boundedness

Let $N = (P, T, F, V, m_0)$ be an eS-Net, m a marking, $p \in P$.

- Let $k \in \mathbb{N}^+$. p is called *k-bounded*, if for each marking m' there holds:

$$m' \in R_N(m_0) \Rightarrow m'(p) \leq k.$$

- p is called *bounded*, if p k -bounded for some $k \in \mathbb{N}^+$.
- N is called *bounded (k-bounded)*, if each place is bounded (k -bounded).
- An eS-net is called *safe*, if it is 1-bounded. Places of a bounded net may be interpreted as boolean conditions.

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Theorem

Let $N = (P, T, F, V, m_0)$ be a eS-Net. N is *unbounded*, i.e. not bounded, iff there exist $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[w \succ m'$ and $m' > m$.

Proof \Leftarrow

Let $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[w \succ m'$ and $m' > m$. It holds

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Thus there must exist at least one unbounded place.

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For each infinite sequence of markings (m_i) of markings there exists an infinite subsequence (m'_j) , which is weakly monotonic, i.e. $l < k$ implies $m'_l \leq m'_k$.

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Proof \Rightarrow

- Consider the reachability graph $EG(N)$, which has an infinite number of nodes. Starting from m_0 there exist a directed path to each node of the graph. Because of the finite number of transitions, each node has only a finite number of direct successors.
- Thus, at m_0 there start an infinite number of paths without cycles, however only a finite number of edges. Therefore, one of these edges must be part of infinitely many paths. Let $m_0 \rightarrow m_1$ be one such edge.
- The same argument can be applied w.r.t. m_1 such that we get $m_0 \rightarrow m_1 \rightarrow m_2$, where $m_1 \rightarrow m_2$ is part of an infinite number of paths.
- The above construction can be repeated infinitely many times. Therefore there exists an infinite sequence of markings (m_i) of pairwise distinct markings, such that m_k, m_l , $0 \leq k \leq l$ implies:

$$m_0[* \succ m_k[* \succ m_l.$$

because of the Lemma there exists an infinite weakly monotonic subsequence (m'_j) von (m_i) . Let m'_1, m'_2 two successive elements. From construction we have $m_0[* \succ m'_1[* \succ m'_2$, $m'_1 \leq m'_2$ and even $m'_1 < m'_2$.

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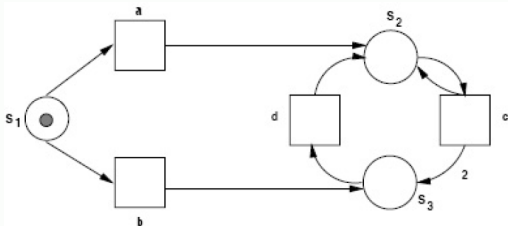
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- If $m \leq m'$, then m' covers m , respectively, m is covered by m' .
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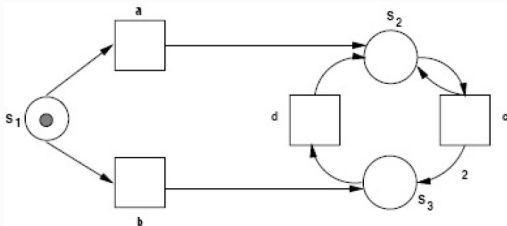
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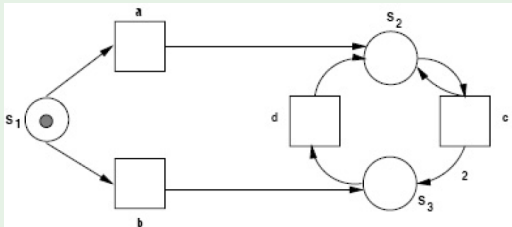
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Coverability Graph

Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Coverability Graph* of N is given by $CG(N) := (R, B)$ as follows:

- *inductive definition of an auxiliary tree $T(N)$:*

The values of the nodes in $T(N)$ are ω -markings of N . The value of the root node r is m_0 . Let m be the value of some node n of $T(N)$, $t \in T$, and $m[t \succ m'$.

- Whenever on the path from the root r to n there exists a node n'' with value m'' such that $m'' < m'$, then update m' by $m'(s) := \omega$ for all places p with $m''(p) < m'(p)$.
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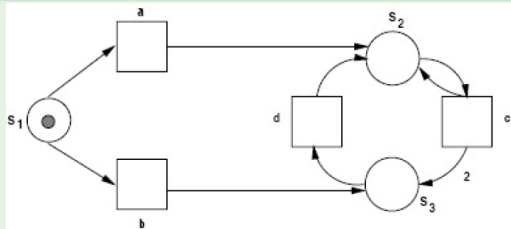
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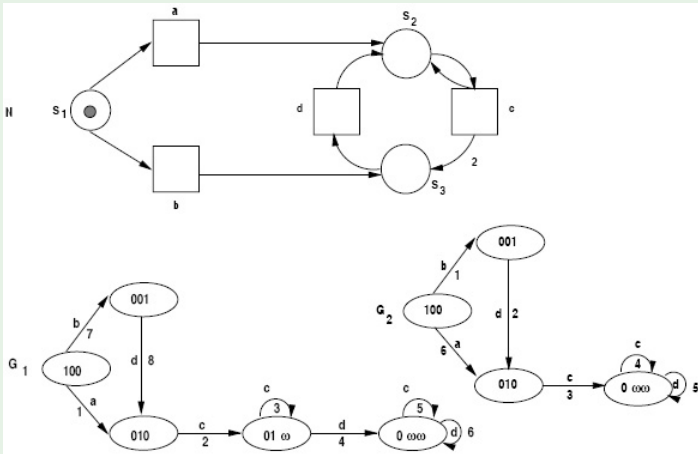
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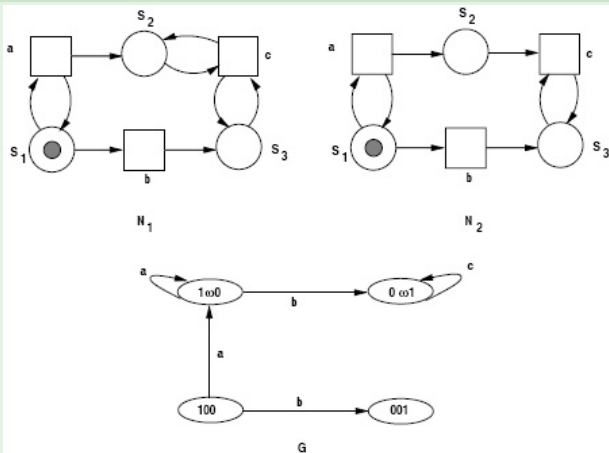
Give a coverability tree.



A eS-net with two different coverability graphs.



Two eS-Nets with identical coverability graphs.



Theorem

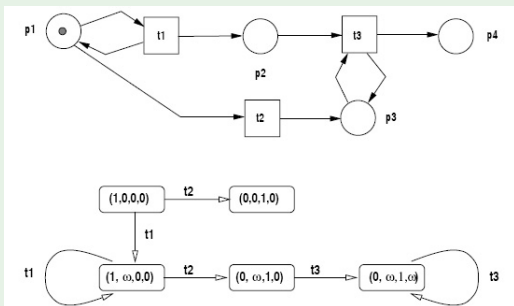
The coverability graph $CG(N) = (R, B)$ of a eS-net N is finite.

Proof:

Assume $CG(N)$ is not finite. Then it contains an infinite number of nodes. Thus there exists an infinite, weakly monotonic sequence of ω -markings, i.e. values of the nodes in the tree. Because of the construction of the auxiliary tree $T(N)$, such an infinite sequence cannot exist, as we can introduce ω only a finite number of times.

To test the reachability of a certain marking we may first test its coverability and then try to find a firing sequence which confirms its reachability.

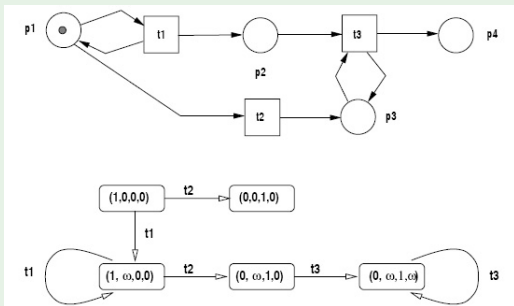
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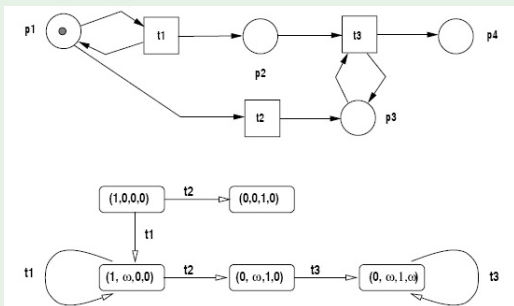
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Live, dead and deadlockfree

Let $N = (P, T, F, V, m_0)$ a eS-Net.

- A marking m is called *dead* in N , if there is no $t \in T$ which is enabled at m .
- A transition t is called *dead* at marking m , if there is no marking reachable from m , such that t is enabled.

If t dead at m_0 , then t is called dead in N .

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Note: whenever a transition is dead at some m , then it is not live at m .

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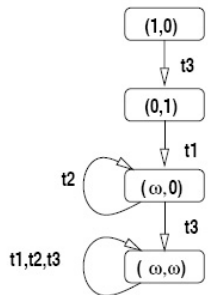
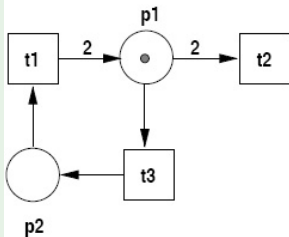
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- A transition t is called *dead* at marking m , if there is no marking reachable from m , such that t is enabled.

If t dead at m_0 , then t is called dead in N .
- A transition t is called *live* at marking m , if for any reachable marking from m it holds that t is not dead.
If $m = m_0$, then t is called *live* in N .
- A marking m is called *live* in N if all transitionen $t \in T$ are *live* in m . If $m = m_0$ then N is called *live*.
- N is called *deadlockfree*, if no dead marking is reachable.

Note: whenever a transition is dead at some m , then it is not live at m .

However, the other direction does not hold.

Firing the word $t_3t_1t_2$ results in a dead marking $(0,0)$. The coverability graph does not indicate this!



Liveness cannot be tested by inspection of the coverability graph.

Do there exist other techniques for analysis?