Chapter 12: Modeling and Analysis of Distributed Applications

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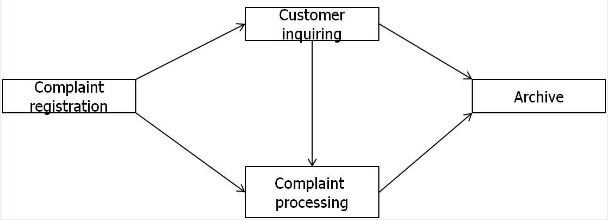
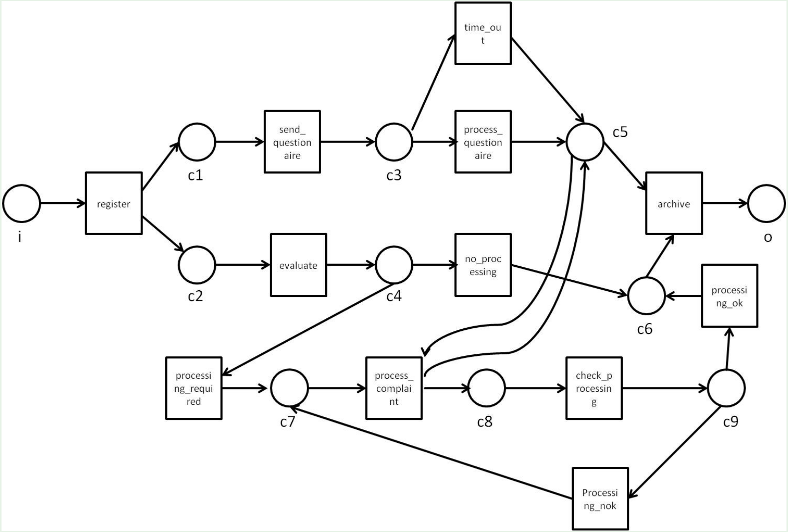
Petri-Nets

- Petri-nets are abstract formal models capturing the flow of information and objects in a way which makes it possible to describe distributed systems and processes at different levels of abstraction in a unified language.
- Petri-nets have the name from their inventor Carl Adam Petri, who introduced this formalism in his PhD-thesis 1962.

Processing of complaints: informal description.

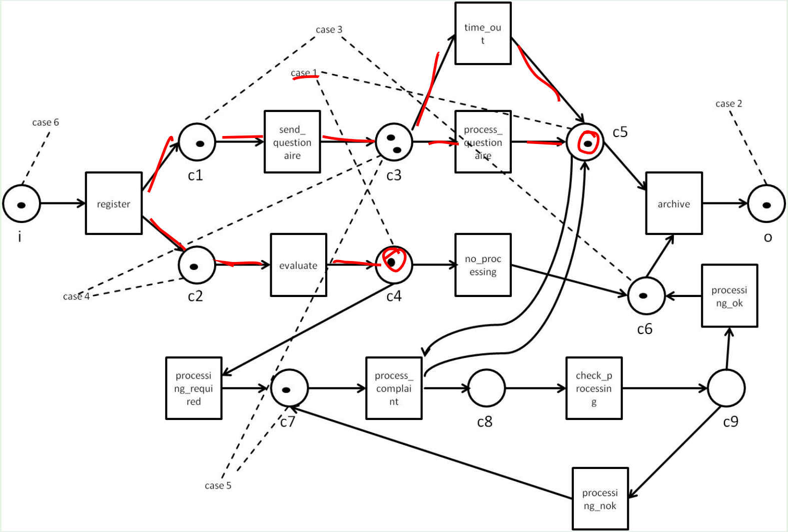


Complaints processing: formal Petri-net orchestration.[1]

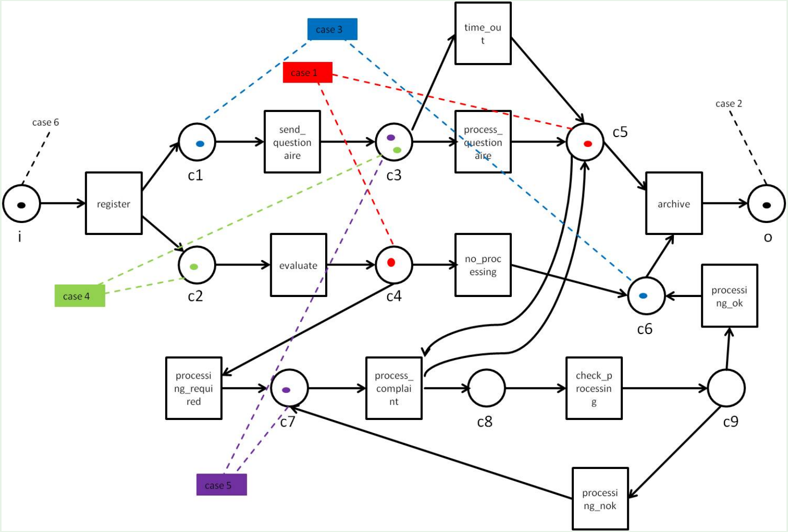


1 van der Aalst: The Application of Petri nets to Workflow Management. Journal of Circuits, Systems, and Computers 8(1): 21-66 (1998)

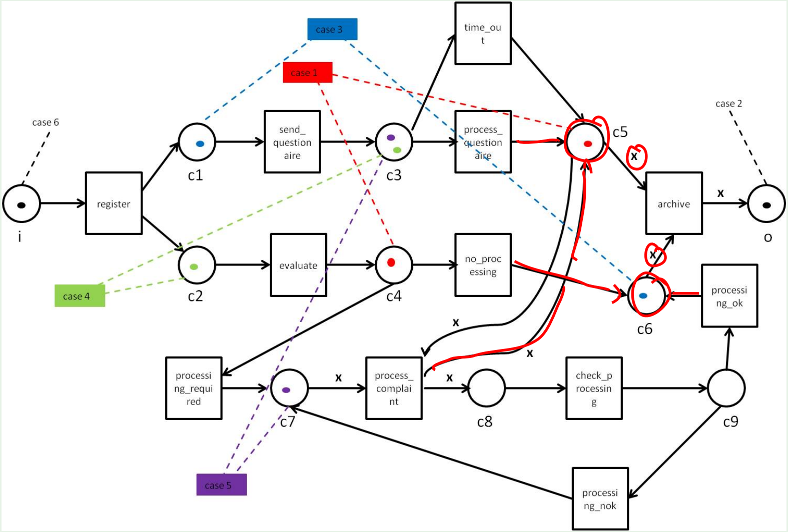
Complaints processing: more than one complaint



Complaints processing: how to distinguish complaints



Complaints processing: keeping things together



Petri-nets

Petri-nets model system dynamics.

- Activities trigger state transitions,
- activities impose control structures,
- applicable for modelling discrete systems.

Benefits

- Uniform language,
- can be used to model sequential, causual independent (concurrent, parallel, nondeterministic) and monitored exclusive activities.
- open for formal analysis, verification and simulation,
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The name *Petri-net* denotes a variety of different versions of nets - we will discuss the special case of *System Nets* following the naming introduced by W. Reisig.

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Section 12.1 Elementary System Nets

UML → sequence ⟷
↘ Activity

automata / sta
m

Basic elements of an elementary System Net (eS-Net)

- System states are represented by *places*, graphically circles or ovals.
- A place may be marked by an arbitrary number of *tokens* graphically represented by black dots.
- System dynamics is represented by *transitions*, graphically rectangles.
- *Transitions* represent activities (events) and the causalities between such activities (events) are represented by edges.
- *Multiplicities* represent the consumption, respectively creation of resources which are caused by the *occurence* of activities.

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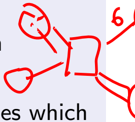
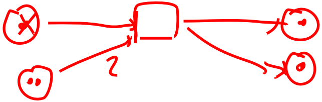
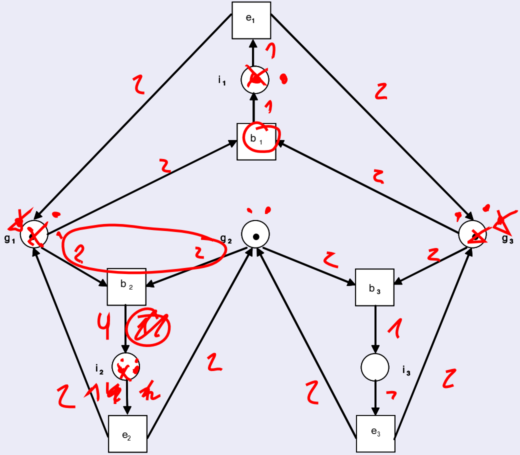
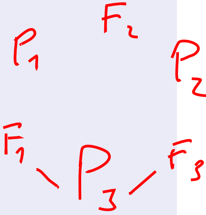




3-Philosopher-Problem

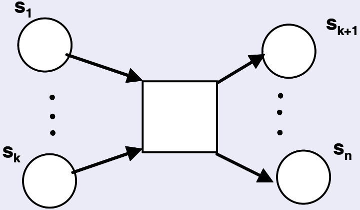
b_j: philosopher starts eating; e_j: philosopher stops eating;
i_j: philosopher is eating; g_j: fork on the desk;
$1 \leq j \leq 3$.





A transition *may* occur when certain conditions with respect to the markings of its directly connected places are fulfilled; the *occurence* of a transition - also called its *firing* - effects the markings of its directly connected edges, i.e. has local effects.

The *surrounding* of a transition t is given by t and all its directly connected places:



$s_1, \ldots, s_k$ are called *preconditions (pre-places)*, $s_{k+1}, \ldots, s_n$ *postconditions (post-places)*.

A place which is pre- and post-place at the same time is called a *loop*.

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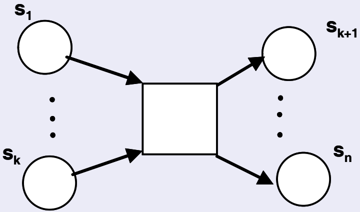

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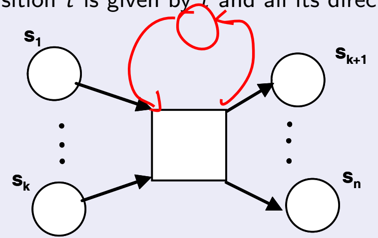

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A *net* is given as a tripel $N = (P, T, F)$, where

- P, the set of *places*, and T, the set of *transitionen*, are non-empty disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$, is the set of directed edges, called *flow relation*, which is a binary relation such that $dom(F) \cup cod(F) = P \cup T$.

~ range

Let $N = (P, T, F)$ be a net and $x \in P \cup T$.

$$xF := \{y \mid (x, y) \in F\}$$
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For $p \in P$, pF is the set of *post-transitions* of p; Fp is the set of *pre-transitions* of p.
For $t \in T$, tF is the set of *post-places* of t; Ft is the set of *pre-places* of t.

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Let $N = (P, T, F)$ be a net. Any mapping m from P into the set of natural numbers NAT is called a *marking* of P.

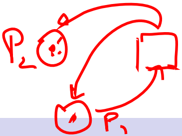
A mapping $P \rightarrow NAT \cup \{\omega\}$ is called ω-*marking*. ω represents an infinitly large number of tokens.

Arithmetic of ω:

$$\omega - n = \omega, \omega + n = \omega, n \cdot \omega = \omega, 0 \cdot \omega = 0, \omega > n$$

where $n \in NAT, n > 0$.

A *marking* represents a possible system state.



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A *eS-Net* is given as $N = (P, T, F, V, m_0)$, where

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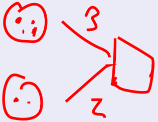
A transition may fire once it is enabled.

Let $N = (P, T, F, V, m_0)$ a eS-Net, m a marking and $t \in T$ a transition.

- *t is enabled at m*, if for all pre-places $p \in Ft$ there holds:

$$m(p) \geq V(p, t).$$



- Whenever t is enabled at m, then t may *fire* at m. Firing t at m transforms m to m', $m[\,t \succ m'$, in the following way:

$$m'(p) := \begin{cases} m(p) - V(p,t) + V(t,p) & \text{if } p \in Ft, p \in tF, \\ m(p) - V(p,t) & \text{if } p \in Ft, p \notin tF, \\ m(p) + V(t,p) & \text{if } p \notin Ft, p \in tF, \\ m(p) & \text{otherwise.} \end{cases}$$

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Transitions and markings in terms of vectors

Let places in P be linearily ordered.

- Markings of a net can be considered as vectors of nonnegative integers of dimension $\mid P \mid$, called *place-vectors*.
- Transitions t can be characterized as vectors of nonnegative integers of dimension $\mid P \mid$, called *transition vectors* $\Delta t, t^+, t^-$:

  Let $N = (P, T, F, V, m_0)$ a eS-Net, $p \in P$ and $t \in T$.

$$t^+(p) := \begin{cases} V(t,p) & \textit{if } p \in tF, \\ 0 & \textit{otherwise.} \end{cases}$$

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Place and transition vectors at work:

- $m \leq m'$, if $m(p) \leq m'(p)$ for $\forall p \in P$,
- $m < m'$, if $m \leq m'$, however $m \neq m'$.

- t is enabled at m iff $t^- \leq m$,
- $m[\, t \succ m'$ iff $t^- \leq m$ and $m' = m + \Delta t$.

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Reachability

Let $N = (S, T, F, V, m_0)$ a eS-Net.

We denote $W(T)$ the set of words with finite length over T; $\epsilon \in W(T)$ is called the *empty word*.

The length of a word $w \in W(T)$ is given by $l(w)$. We have $l(\epsilon) = 0$.

Let m, m' be markings of P and $w \in W(T)$. We define a relation $m[\, w \succ m'$ inductively:

- $m[\, \epsilon \succ m'$ iff $m = m'$,
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- $L_N(m) := \{w \mid \exists m' : m[w\succ m'\}$, the set of all words representing firing sequences of transitions of N starting at m,
- $\Delta w := \sum_{i=1}^{n} \Delta t_i$, wobei $w = t_1 t_2 \dots t_n$.

Results

- $[*\succ$ is reflexiv and transitiv.
- $m[w\succ m' \Rightarrow (m + m^*)[w\succ(m' + m^*), \forall m^* \in NAT^{|S|}$. (Monotonie)
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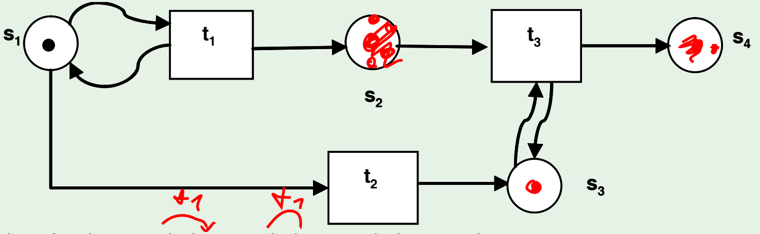
- $m[\,w\succ m' \Rightarrow (m + m^*)[\,w\succ(m' + m^*), \forall m^* \in NAT^{|S|}$. (Monotonie)
- $m[\,w\succ m' \Rightarrow m' = m + \Delta w$.

Reachability graph

Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Reachability graph* of N is a directed graph $EG(N) := (R_N(m_0), B_N)$; $R_N(m_0)$ is the set of nodes and B_N is the set of annotated edges as follows:

$$B_N = \{(m, t, m') \mid m, m' \in R_N(m_0), t \in T, m[t \succ m'\}.$$

Exercise: Give the reachability graph of the following eS-Net:



$$R_N(m_0) = \{ \quad (1,0,0,0),(1,1,0,0),(1,2,0,0),(1,3,0,0),\dots,$$
$$(0,0,1,0),(0,1,1,0),(0,2,1,0),(0,3,1,0),\dots,$$
$$(0,0,1,1),(0,1,1,1),(0,0,1,2),(0,2,1,1),(0,1,1,2),(0,0,1,3),\dots\}$$

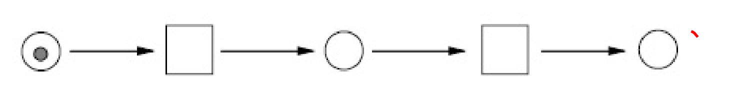
$$L_N(m_0) = \{ \quad \epsilon, t_1, t_1t_1, t_1t_1t_1, \dots,$$
$$t_2, t_1t_2, t_1t_1t_2, t_1t_1t_1t_2, \dots,$$
$$t_1t_2t_3, t_1t_1t_2t_3, t_1t_1t_2t_3t_3, t_1t_1t_1t_2t_3, t_1t_1t_1t_2t_3t_3, t_1t_1t_1t_2t_3t_3t_3, \dots\}$$

Section 12.2 Control Patterns

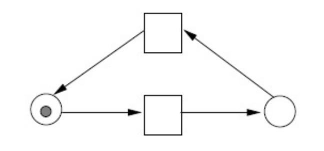
- eS-nets can be used to model *causal dependencies*; for modelling temporal aspects extensions of the formalism are required.
- Whenever between some transitions there are no causal dependencies, the transitions are called *concurrent*; concurrency is a prerequisite for parallelism.

Some typical causalities

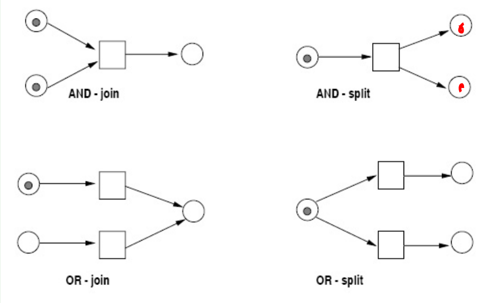
Sequence



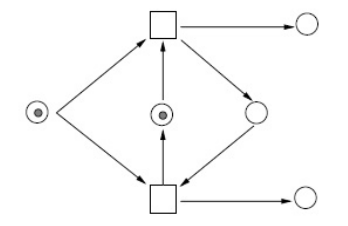
Iteration



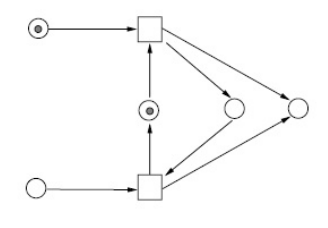
AND-join, OR-join, AND-split, OR-split



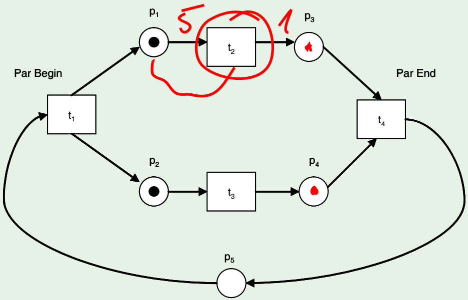
OR-Split with regulation



OR-Join with regulation



A eS-Net with concurrency



Section 12.3 Analysis

Boundedness

Let $N = (P, T, F, V, m_0)$ be a eS-Net, m a marking, $p \in P$.

- Let $k \in NAT^+$. p is called *k-bounded*, if for each marking m' there holds:

$$m' \in R_N(m_0) \Rightarrow m'(p) \leq k.$$

- p is called *bounded*, if p k-bounded for some $k \in NAT^+$.
- N is called *bounded (k-bounded)*, if each place is bounded (k-bounded).
- A eS-net is called *safe*, if it is 1-bounded. Places of a bounded net may be interpreted as boolean conditions.

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Theorem

Let $N = (P, T, F, V, m_0)$ be a eS-Net. N is *unbounded*, i.e. not bounded, iff there exist $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[\,w\succ m'$ and $m' > m$.

Proof $\Leftarrow$

Let $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[\,w\succ m'$ and $m' > m$. It holds

$$m[\,w\succ m'[\,w\succ m''[\,w\succ m''' \ldots,$$

where $m < m' < m'' < m''' < \ldots$.

Thus there must exist at least one unbounded place.

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To proof $\Rightarrow$ we first proof:

Lemma

For each infinite sequence of markings (m_i) of markings there exists an infinite subsequencev (m'_j), which is weakly monotonic, i.e. $l < k$ implies $m'_l \leq m'_k$.

To prove the Lemma, first extract an infinite subsequence for which weak monotonicity holds for the first components of its markings. Then extract from that subsequence an infinite subsequence for which weak monotonicity holds for the second components of its markings, etc.

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Proof $\Rightarrow$

- Consider the reachability graph $EG(N)$, which has an infinite number of nodes. Starting from m_0 there exist a directed path to each node of the graph. Because of the finite number of transitions, each node has only a finite number of direct successors.
- Thus, at m_0 there start an infinite number of paths without cycles, however only a finite number of edges. Therefore, one of these edges must be part of infinitly many paths. Let $m_0 \rightarrow m_1$ be one such edge.
- The same argument can be applied w.r.t. m_1 such that we get $m_0 \rightarrow m_1 \rightarrow m_2$, where $m_1 \rightarrow m_2$ is part of an infinite number of paths.
- The above construction can be repeated infinitly many times. Therefore there exists an infinite sequence of markings (m_i) of pairwise distinct markings, such that m_k, m_l, $0 \leq k \leq l$ implies:
$$m_0[*\succ m_k[*\succ m_l.$$
because of the Lemma there exists an infinite weakly monotonic subsequence (m'_j) von (m_i). Let m'_1, m'_2 two successive elements. From construction we have $m_0[*\succ m'_1[*\succ m'_2$, $m'_1 \leq m'_2$ and even $m'_1 < m'_2$.

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Reachability

Let $N = (P, T, F, V, m_0)$ be a eS-Net, $m \in NAT^{|P|}$ a marking. The decision problem:

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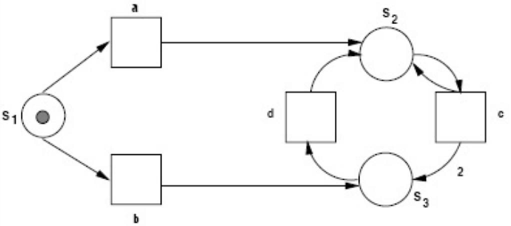
Coverability

Let $N = (P, T, F, V, m_0)$ be a eS-Net and let m, m' be markings of N.

- If $m \leq m'$, then m' *covers* m, respectively, m *is covered by* m'.
- m is called *coverable* in N, if there exists a reachable marking m' which covers m.

Consequence: Whenever a marking is not coverable w.r.t. some eS-Net N, it is not reachable in N.

Give examples.



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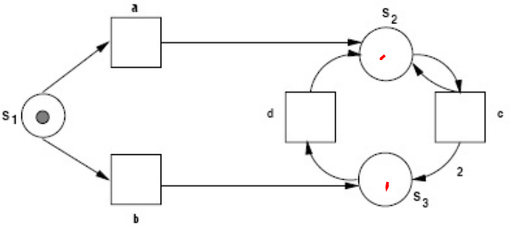

(1,0,0)

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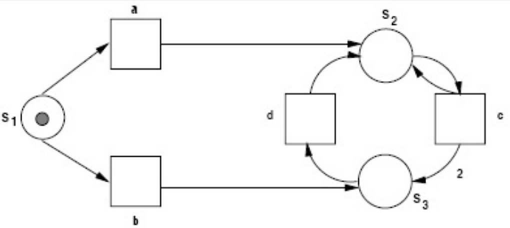

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Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Coverability Graph* of N is given by $CG(N) := (R, B)$ as follows:

- *inductive definition of an auxiliary tree $T(N)$:*

  The values of the nodes in $T(N)$ are ω-markings of N. The value of the root node r is m_0. Let m be the value of some node n of $T(N)$, $t \in T$, and $m[t\succ m'$.
  - Whenever on the path from the root r to n there exists a node n'' with value m'' such that $m'' < m'$, then update m' by $m'(s) := \omega$ for all places p with $m''(p) < m'(p)$.
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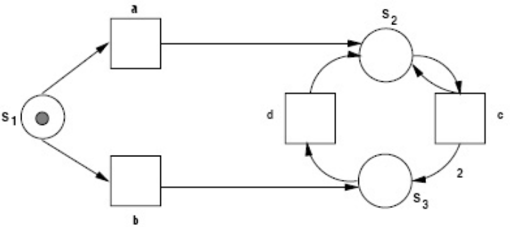
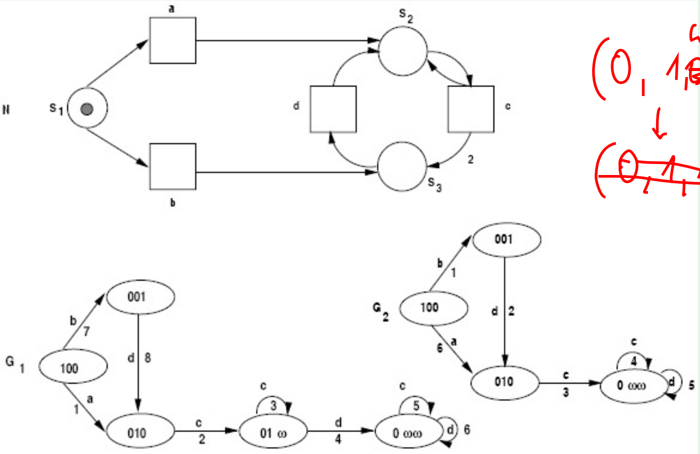
Give a coverability tree.



A eS-net with two different coverability graphs.

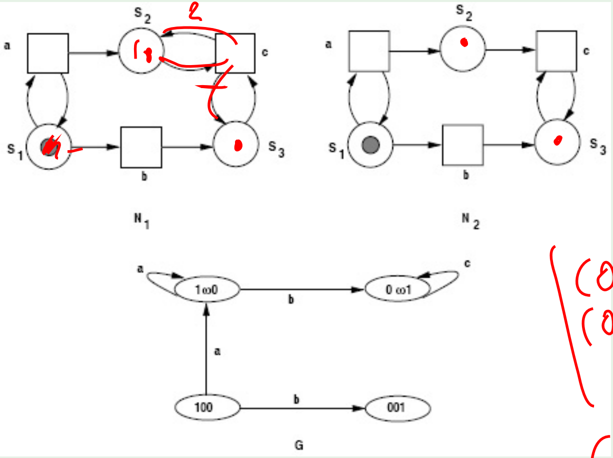


$(0, 1\omega)$ $(0, 1$

~~$(0, 1, 1)$~~ $(0, 2$

$(0,$

Two eS-Nets with identical coverability graphs.



(0, S0,

(0,

(0 1 1) → (0,
(0 2 1| → (0
:
:
(0 ω 1)

Theorem

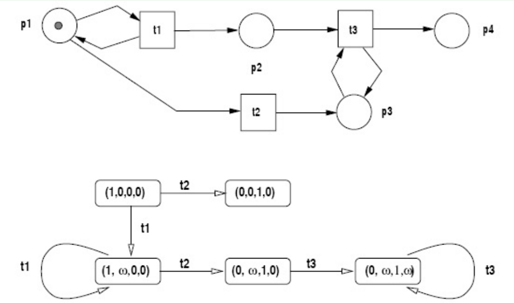
The coverability graph $CG(N) = (R, B)$ of a eS-net N is finite.

Proof:

Assume $CG(N)$ is not finite. Then it contains an infinite number of nodes. Thus there exists an infinite, weakly monotonic sequence of ω-markings, i.e. values of the nodes in the tree. Because of the construction of the auxiliary tree $T(N)$, such an infinite sequence cannot exist, as we can introduce ω only a finite number of times.

To test the reachability of a certain marking we may first test its coverability and then try to find a firing sequence which confirms its reachability.

Is marking $m = (0, 3, 1, 3)$ reachable?



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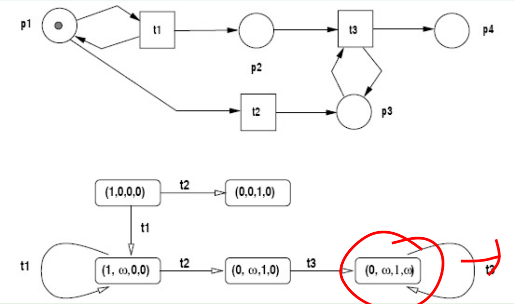

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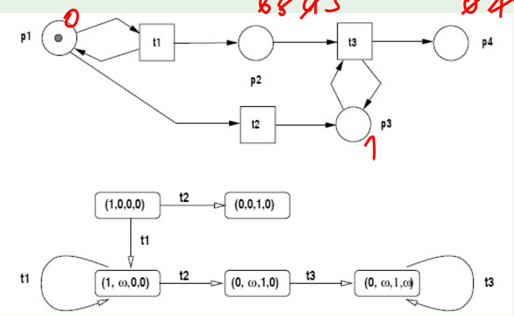

0,0,0 →

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1, 6, 0 0 → 0, 6, 1, 0 $\xrightarrow{t_2}$

Live, dead and deadlockfree

Let $N = (P, T, F, V, m_0)$ a eS-Net.

- A marking m is called *dead* in N, if there is no $t \in T$ which is enabled at m.
- A transition t is called *dead* at marking m, if there is no marking reachable from m, such that t is enabled.

  If t dead at m_0, then t is called dead in N.
- A transition t is called *live* at marking m, if for any reachable marking from m it holds that t is not dead.
  If $m = m_0$, then t is called *live* in N.
- A marking m is called *live* in N if all transitionen $t \in T$ are *live* in m. If $m = m_0$ then N is called *live*.
- N is called *deadlockfree*, if no dead marking is reachable.

Note: whenever a transition is dead at some m, then it is not live at m.

However, the other direction does not hold.

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Note: whenever a transition is dead at some m, then it is not live at m.

However, the other direction does not hold.

Live, dead and deadlockfree

Let $N = (P, T, F, V, m_0)$ a eS-Net.

- A marking m is called *dead* in N, if there is no $t \in T$ which is enabled at m.
- A transition t is called *dead* at marking m, if there is no marking reachable from m, such that t is enabled.

  If t dead at m_0, then t is called dead in N.
- A transition t is called *live* at marking m, if for any reachable marking from m it holds that t is not dead.
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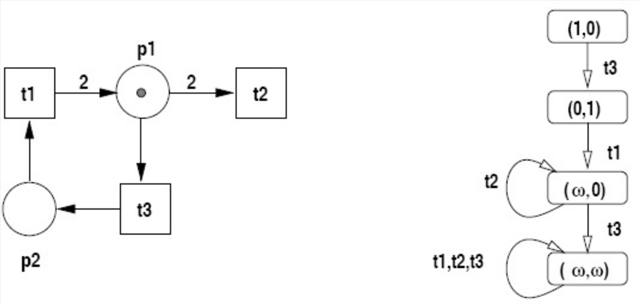
However, the other direction does not hold.

Firing the word $t_3 t_1 t_2$ results in a dead marking $(0, 0)$. The coverability graph does not indicate this!

(1, 0)
(0, 1)
(0, 0)



Lifeness cannnot be tested by inspection of the coverability graph.

Do there exist other techniques for analysis?