Energy Informatics

03 Network Algorithms

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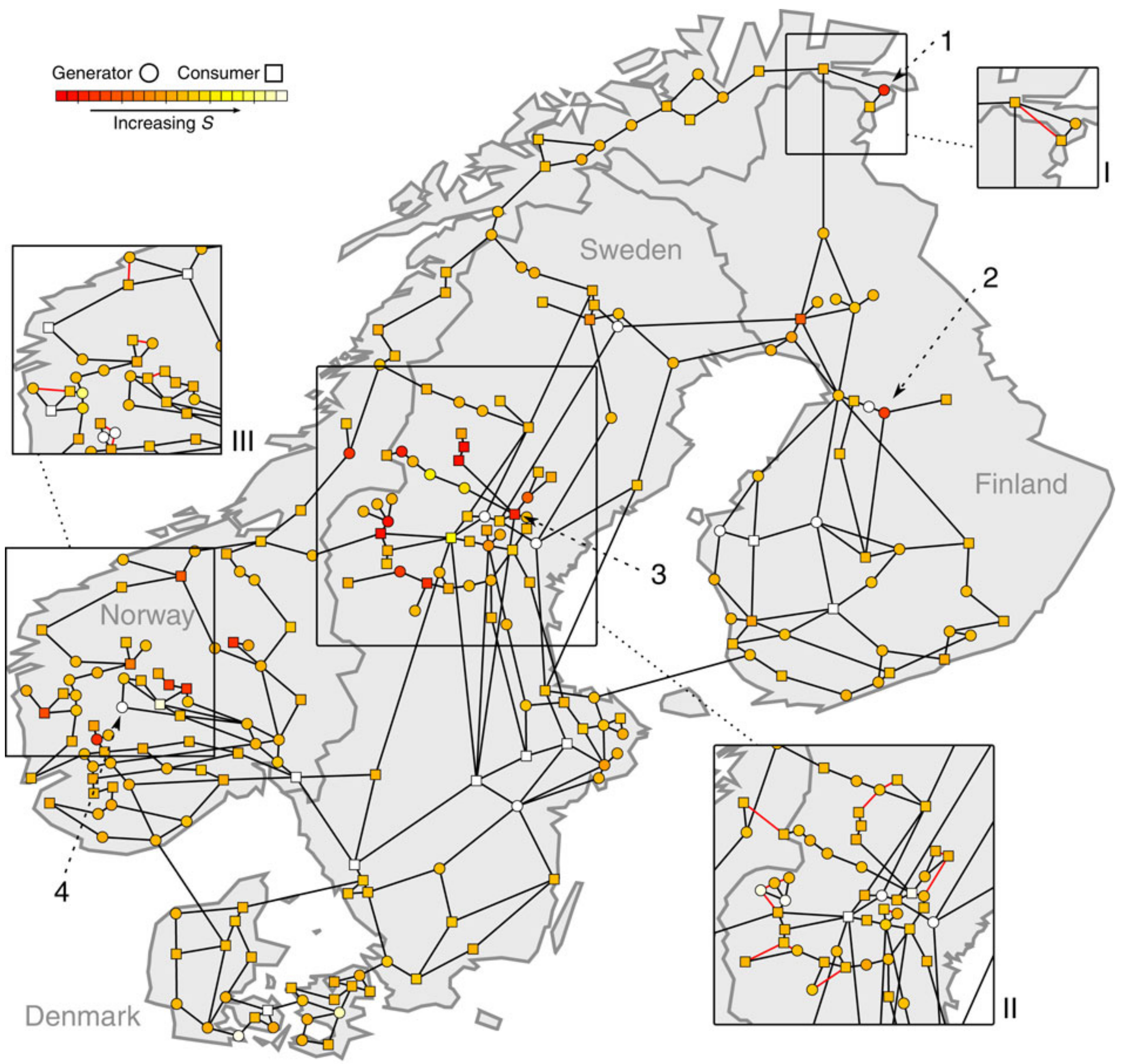
Graph Theory in a Nutshell

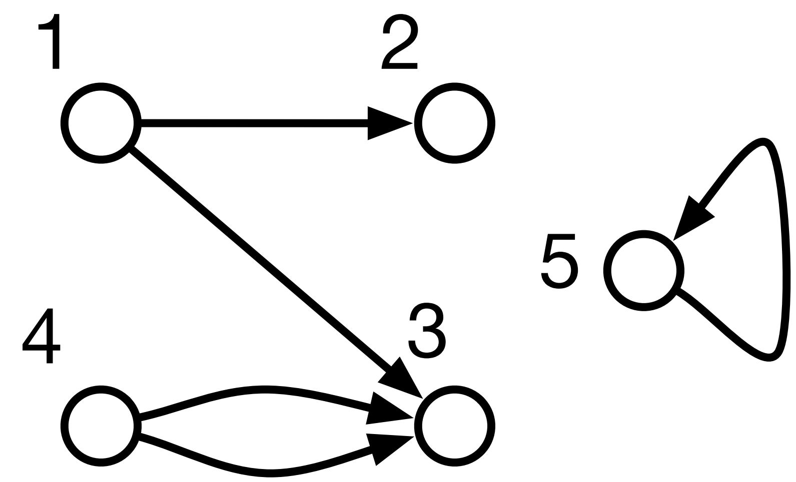
- A graph G=(V,E)
  - nodes/vertices V
  - edges E
    - connecting two nodes
- Variants
  - undirected/directed edge = lines/arrows
  - loops
    - edge connecting the node with itself
  - node/edge weights
    - mapping of numbers to the nodes/edges

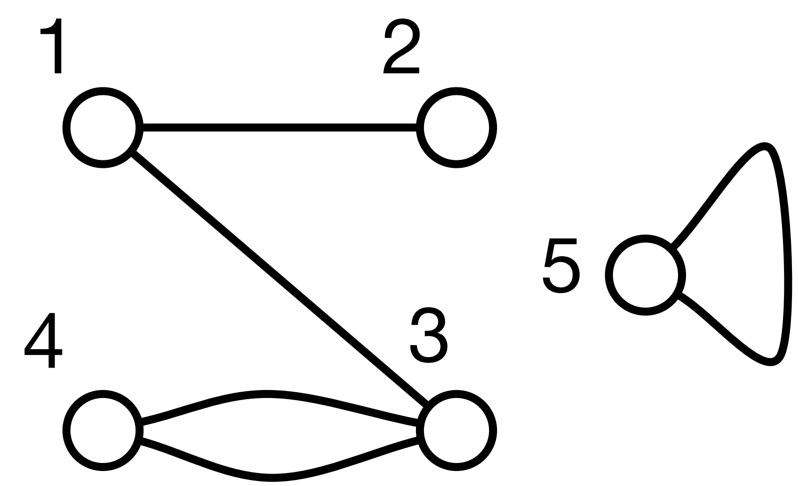


How dead ends undermine power grid stability
Peter J. Menck, Jobst Heitzig, Jürgen Kurths & Hans Joachim Schellnhuber
Nature Communications 5, Article number: 3969

Terms in Graphs

- Degree of a node
  - number of edges at a node u
- Regular graph
  - if the maximum degree = minimum degree in a graph
- Indegree/Out-degree
  - in case of directed graph (digraph), the number of edges pointing to/from a node u
- Two nodes u,v are **adjacent**
  - if they are connected via an edge
- A graph is simple, if there are no loops or no parallel edges
  - usually only simple graphs are considered

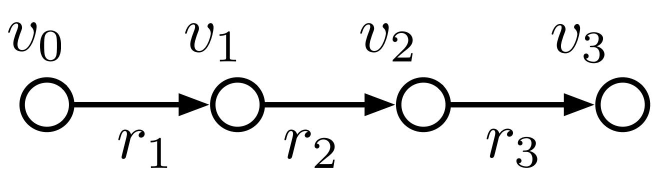




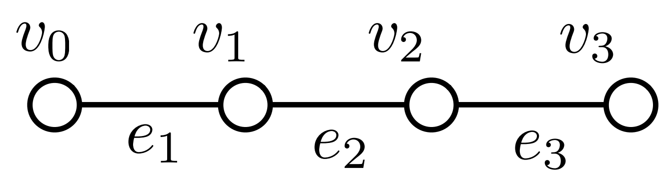
Paths

- A sequence of adjacent nodes is called a path
- Paths with same start and end are called cycles
- The length of a path is the number of edges passed
- A path is simple if no edge occurs twice
  - it is elementary if no node occurs twice

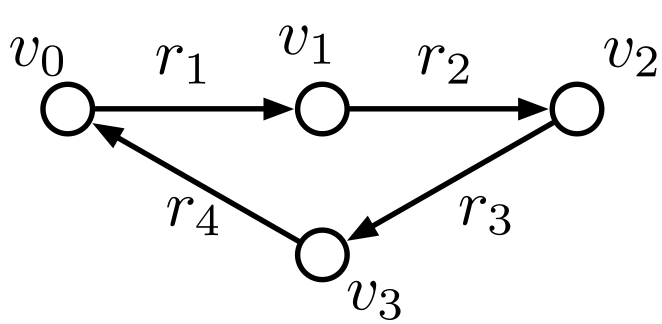
path of length 3



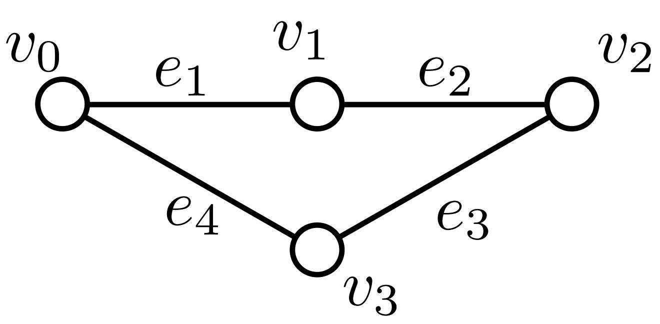
path of length 3



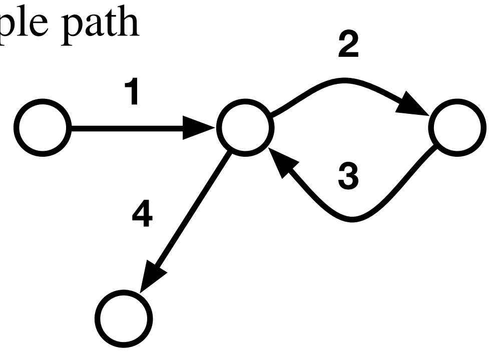
cycle of length 4



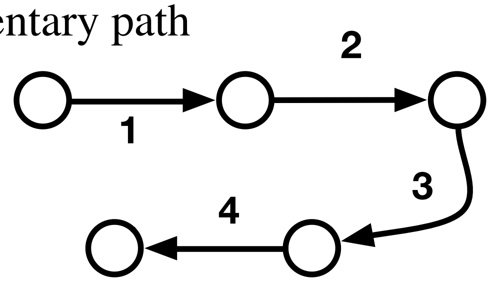
cycle of length 4



walk
simple path

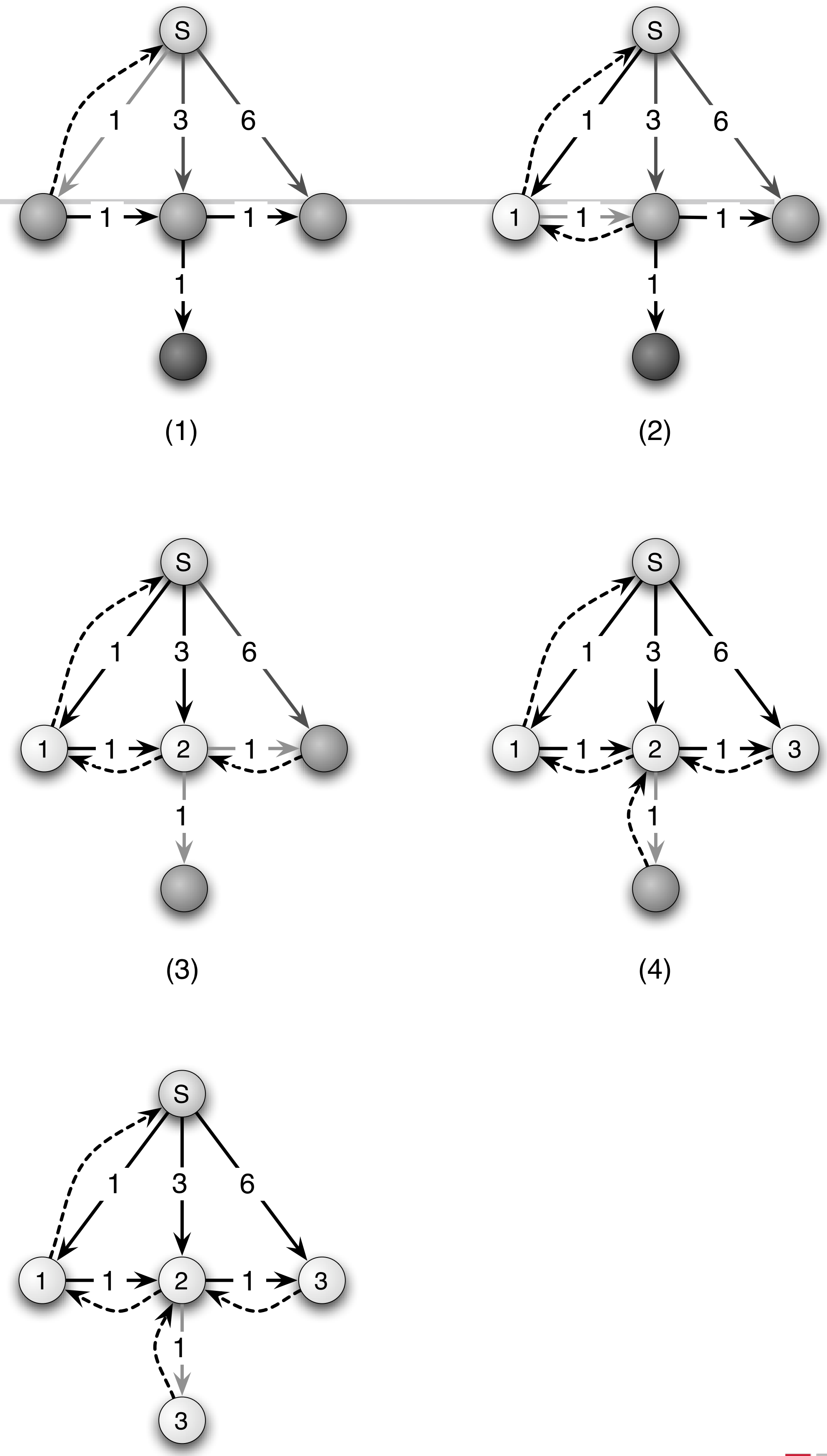


trail
elementary path



Shortest Paths

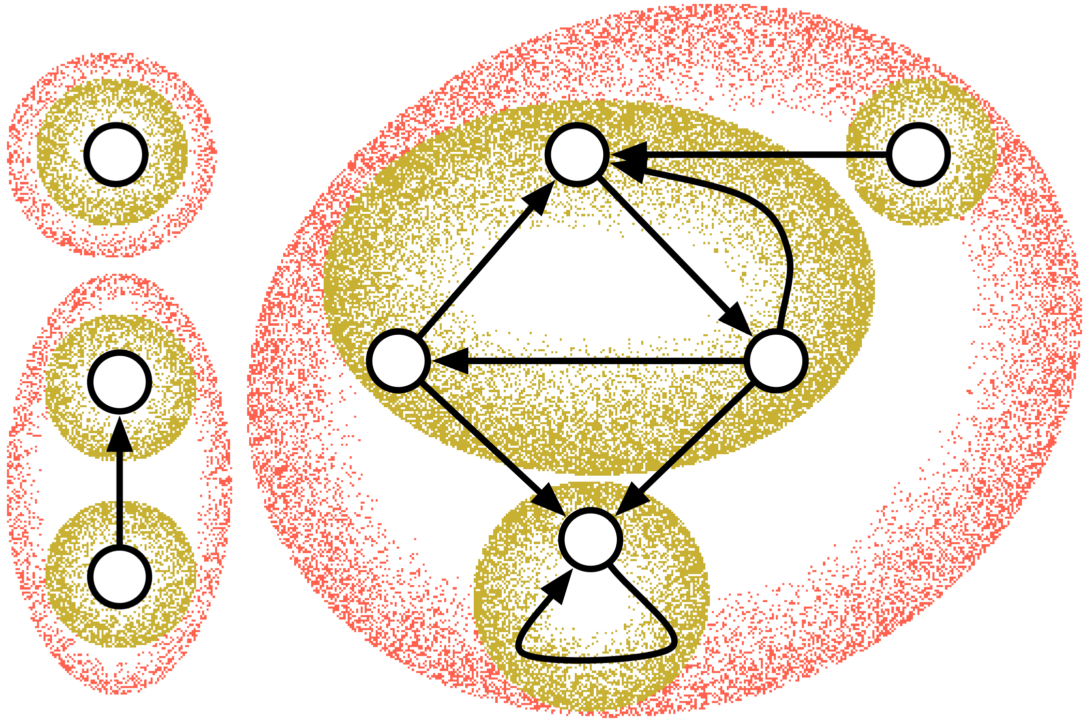
- Given
  - a graph G=(V,E)
  - start node s, target node t
- Compute the shortest path
- Dijkstras algorihm
  - Start with set S={s}
  - In each round
    - add the node u of the neighborhood of S
      - which has the shortest distance to s
    - store the edge used to u

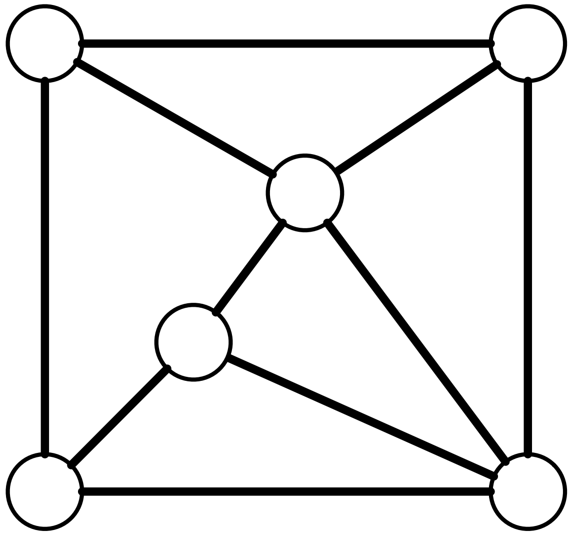


(1) (2) (3) (4) (5)

Connectivity

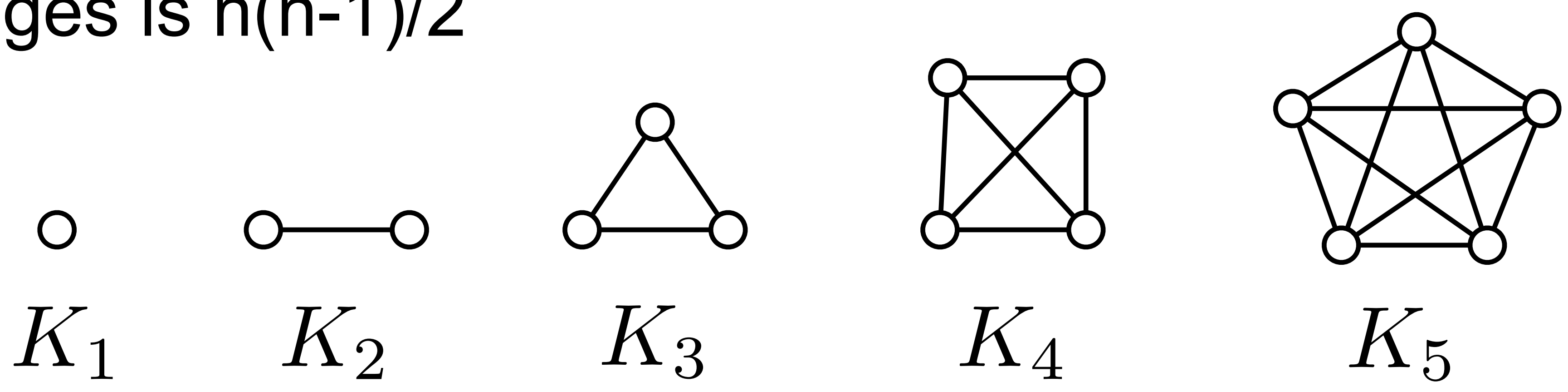
- An undirected graph is **connected**
  - if for all nodes u,v there exists a path connecting u and v
- A directed graph is **weakly connected**
  - if the corresponding undirected graph is connected
- A directed graph is **strongly connected**
  - if for all nodes u,v there exists a directed path connecting u and v
- A graph is **k-(vertex)-connected**, if there are k node disjoint paths between all nodes
- analog definition for d edge connected



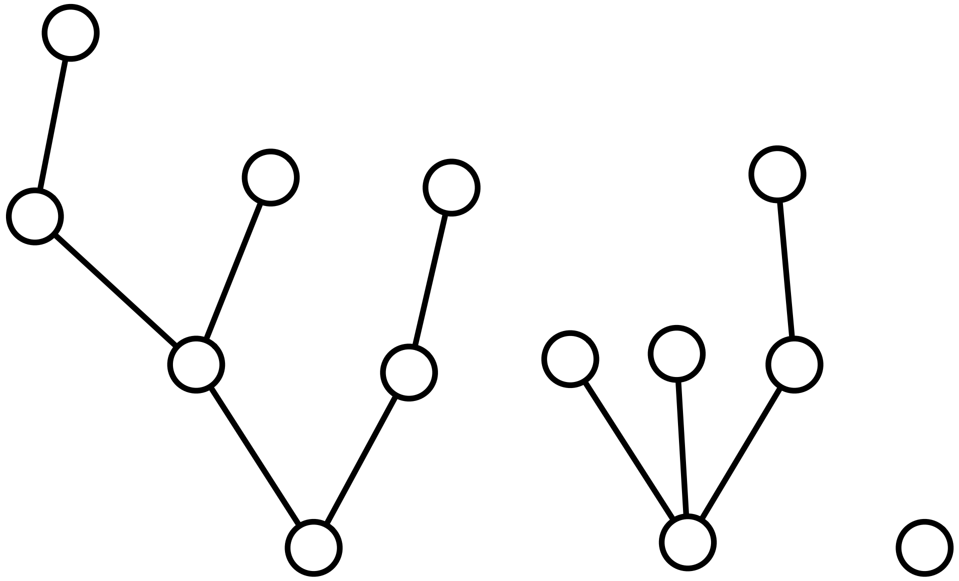


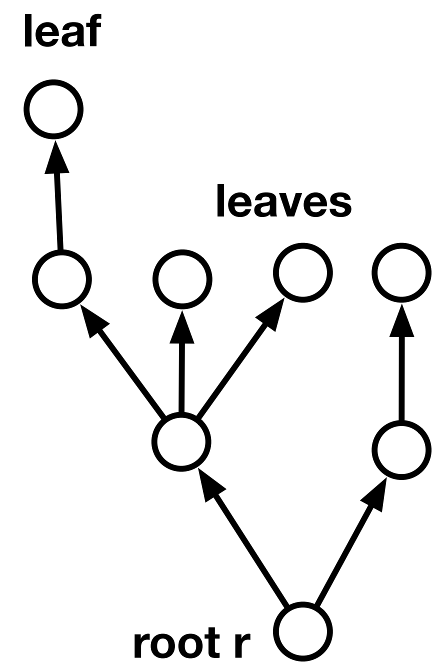
Special Graphs

- Complete (simple) undirected graphs
  - no. of edges is n(n-1)/2



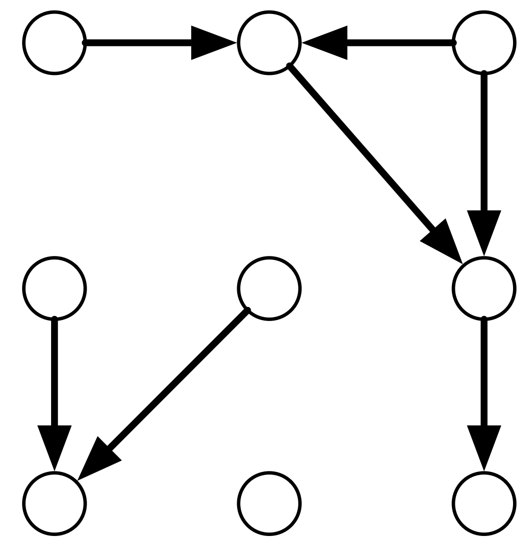
- Trees are connected undirected graphs without cycles
  - sets of graphs are called forests



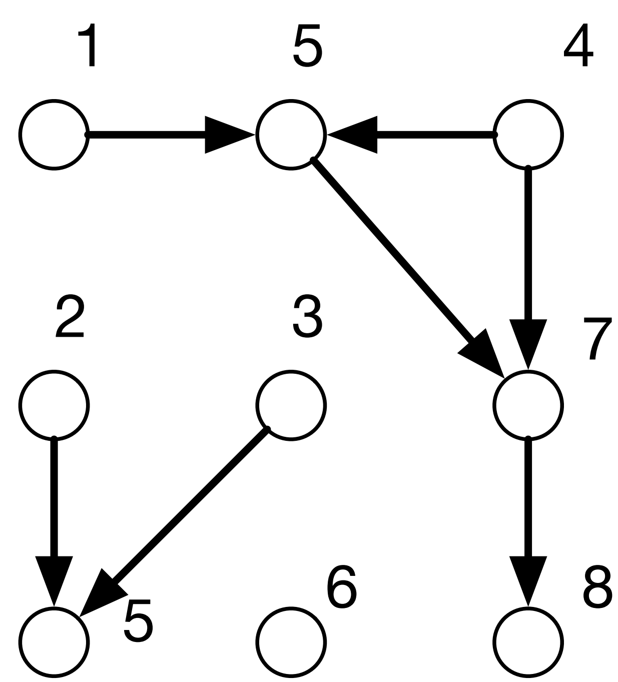


DAG: Directed Acyclic Graph

- Directed acyclic graph

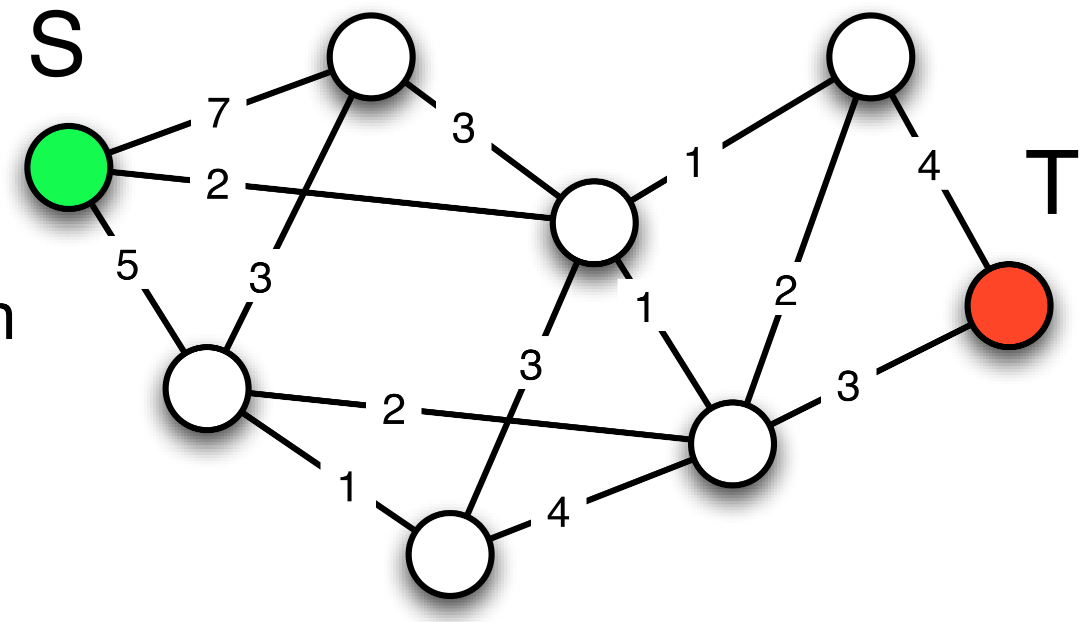


- Topologic Sorting
  - mapping f of {1,..,n} to V such that for edge (u,v)
    - f(u) < f(v)



Flows in Networks

- Motivation
  - Optimize flow from source to target
- Definition:
  - (Single-commodity) maximum flow problem
  - Given
    - a graph G=(V,E)
    - a capacity function $w: E \rightarrow R^{+}_{0}$,
    - source set S and target set T
  - Find a maximum flow from S to T

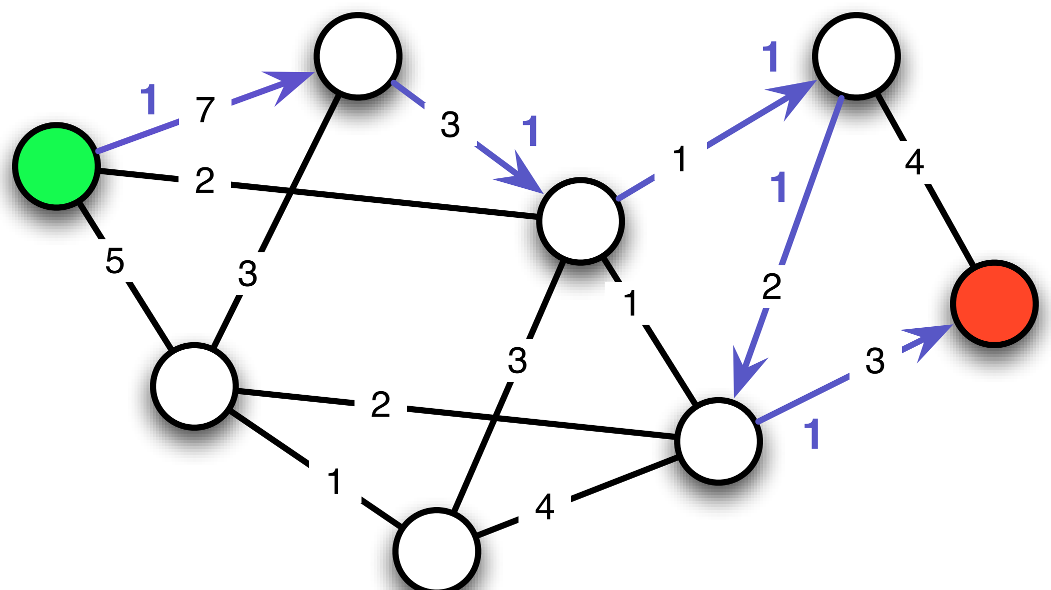


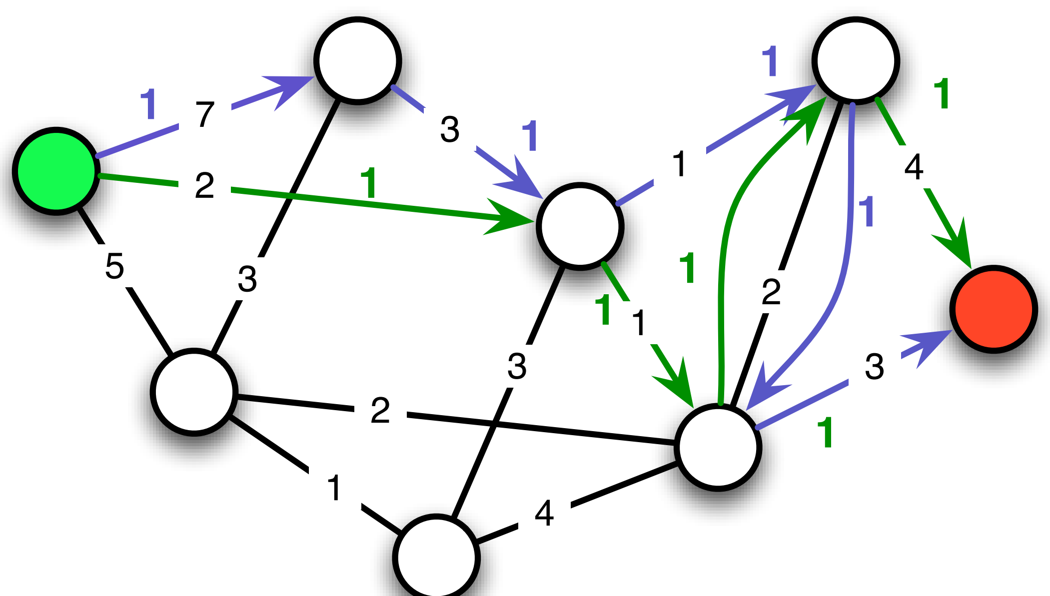
- A flow is a function $f : E \rightarrow R_0^+$ such that
  - for all $e \in E$: $f(e) \leq w(e)$
  - for all $e \notin E$: $f(e) = 0$
  - for all $u,v \in V$: $f(u,v) \geq 0$

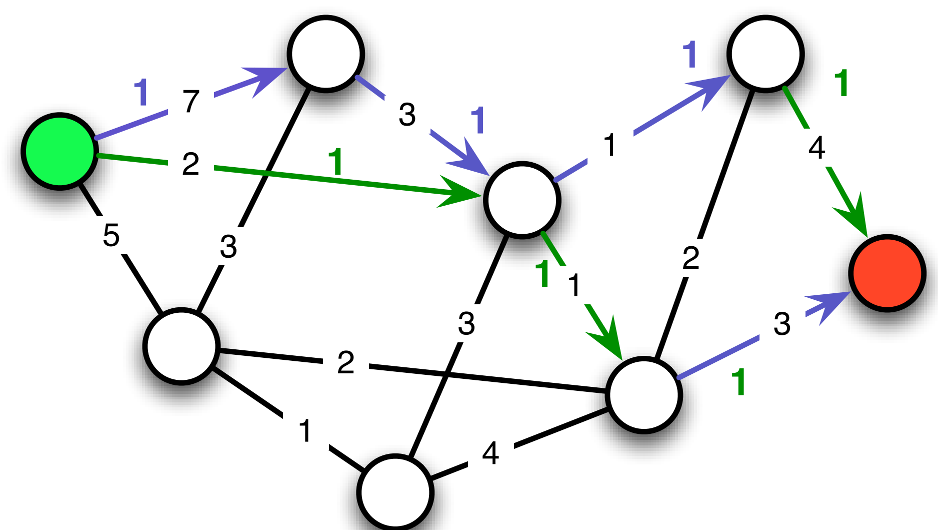
$$\forall u \in V \setminus (S \cup T) \quad \sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$$

- Maximize flow $$\sum_{u \in S} \sum_{v \in V} f(u,v)$$

Flows in Networks

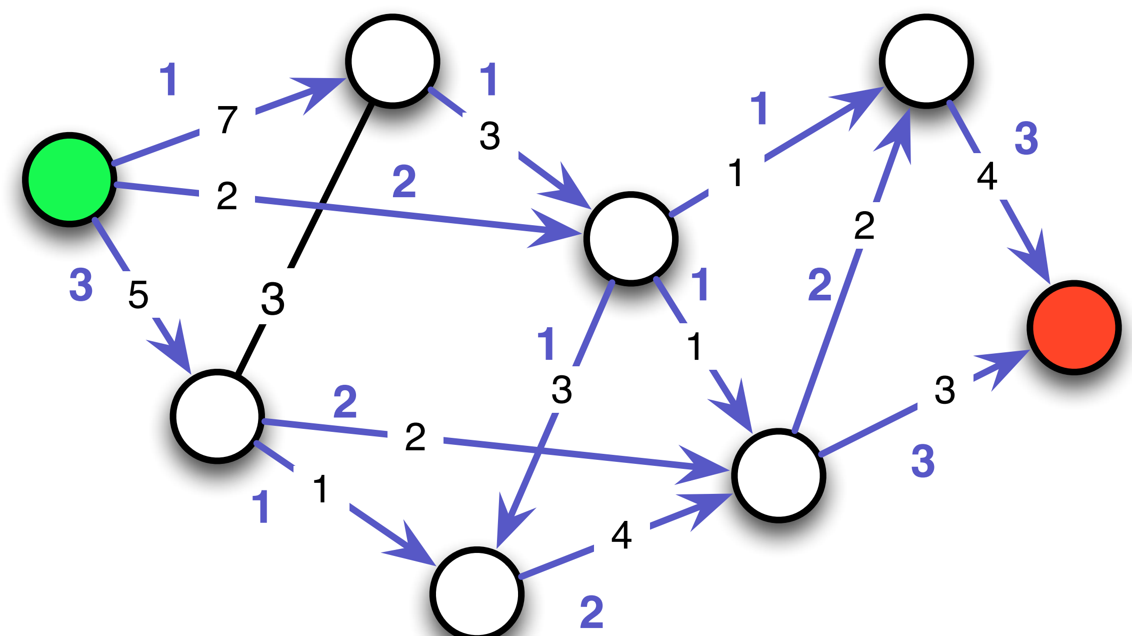






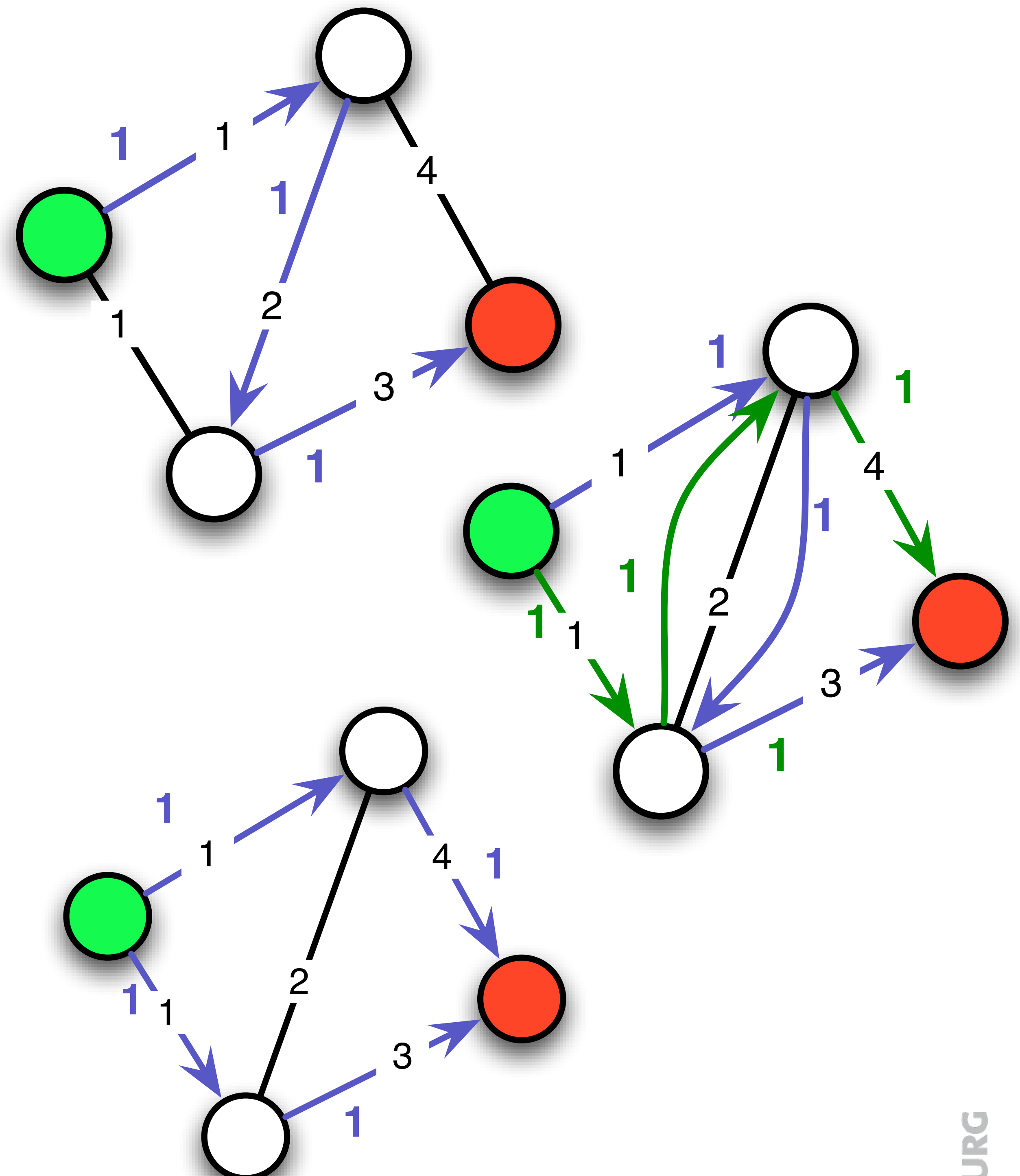
Computation of the Maximum Flow

- Every natural pipe system solves the maximum flow problem
- Algorithms
  - Linear Programming
    - for real numbers
    - the flow is described by equations of a linear optimization problem
    - Simplex algorithm (or Ellipsoid method) can solve any linear equation system
  - Ford-Fulkerson
    - also for integers
    - as long as open paths exist, increase the flow on theses paths
      - open path: path which increases the flow
  - Edmonds-Karp
    - special case of Ford-Fulkerson
    - use BFS (breadth first search) to find open paths



Ford-Fulkerson

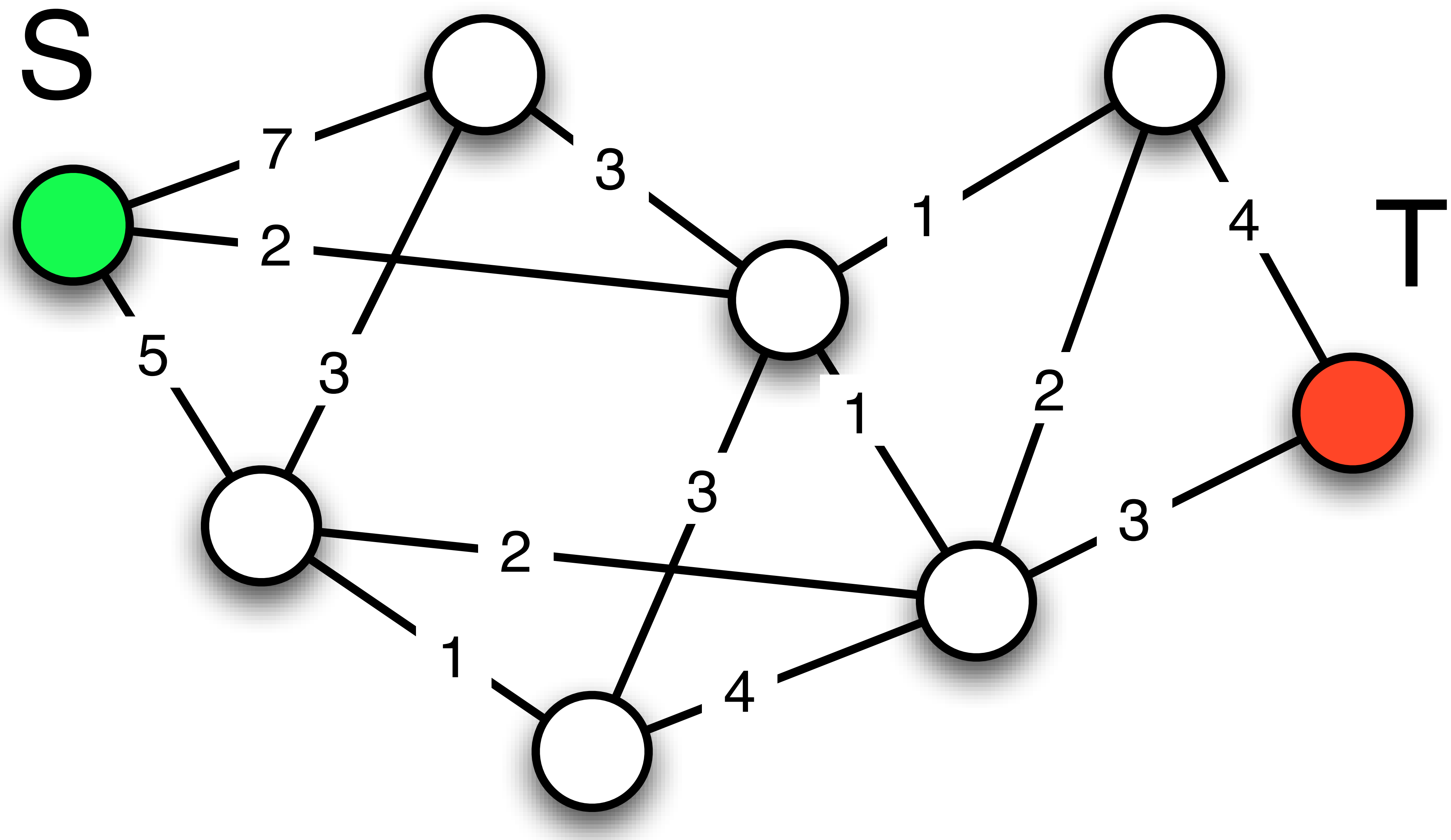
- Find a path from the source node to the target node
  - where the capacity is not fully utilized
  - or which reduces the existing flow
- Compute the maximum flow on this augmenting path
  - by the minimum of the flow that can be added on all paths
- Add the flow on the path to the existing flow
- Repeat this step until no flow can be added anymore



Edmunds-Karp

- **Search path for Ford-Fulkerson algorithm**
- **Choose the shortest augmenting path**
  - Computation by breadth-first-search
- **leads to run-time $O(|V|\,|E|^2)$**
  - whereas Ford-Fulkerson could have exponential run-time

Example



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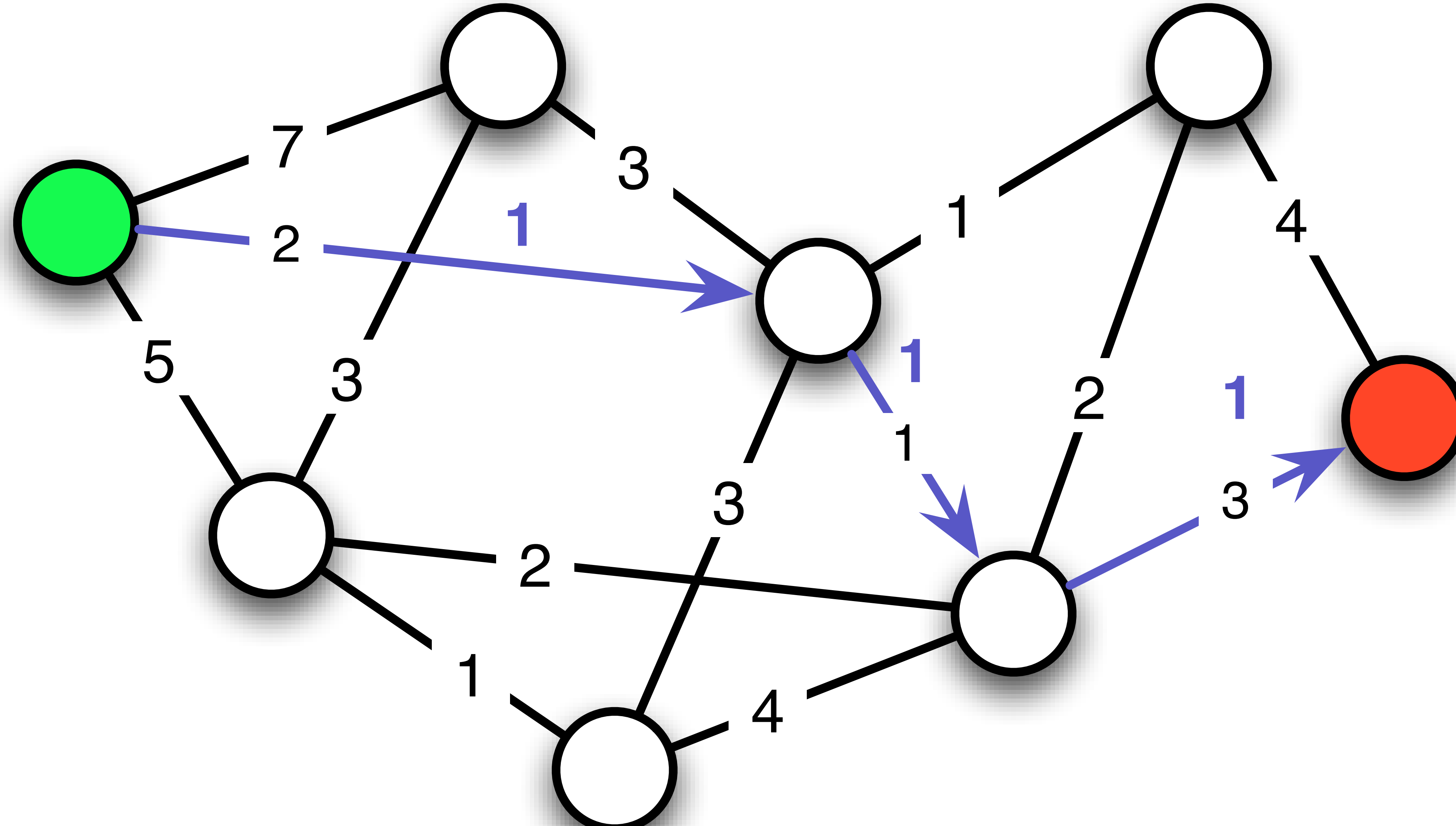

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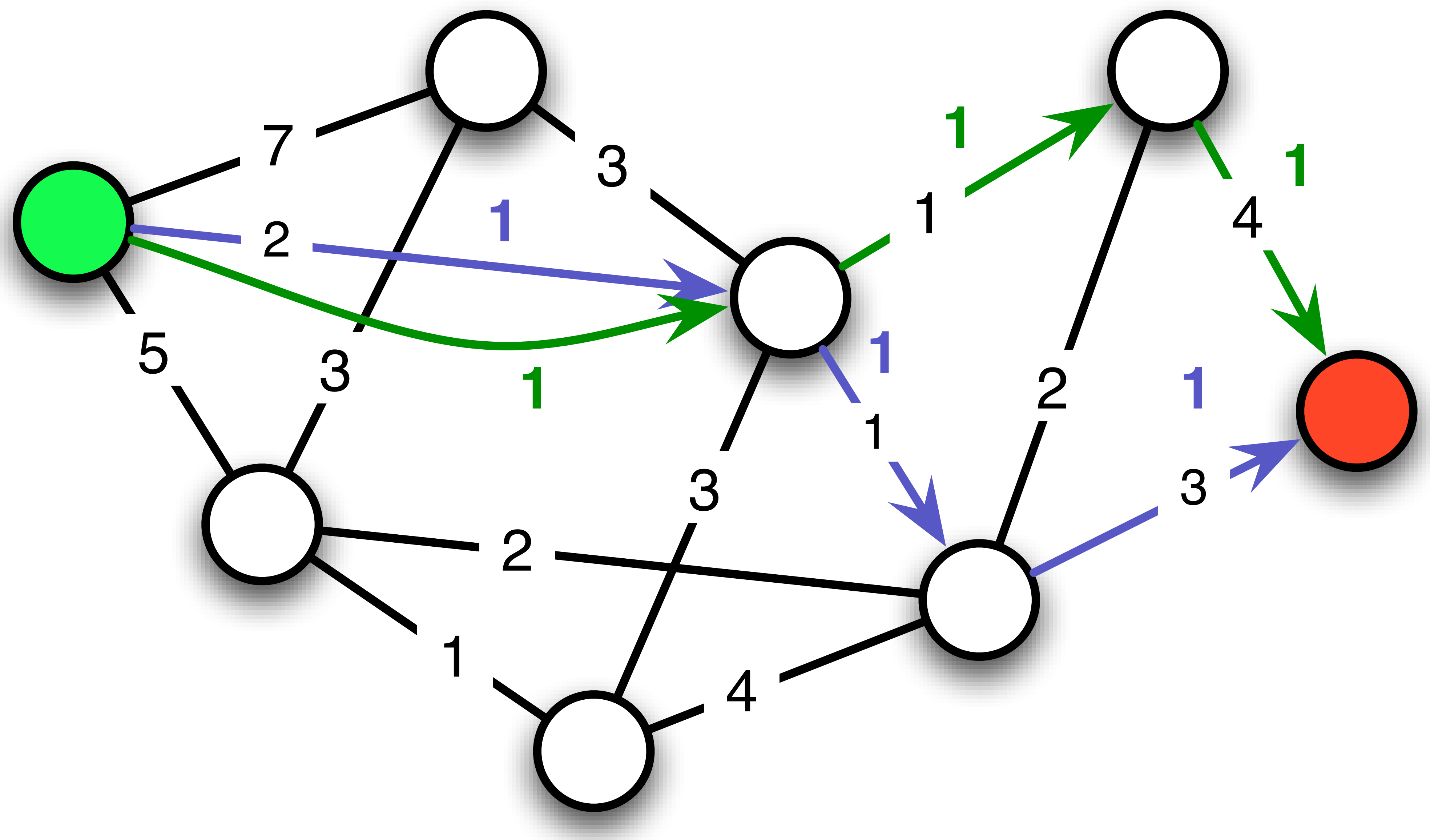

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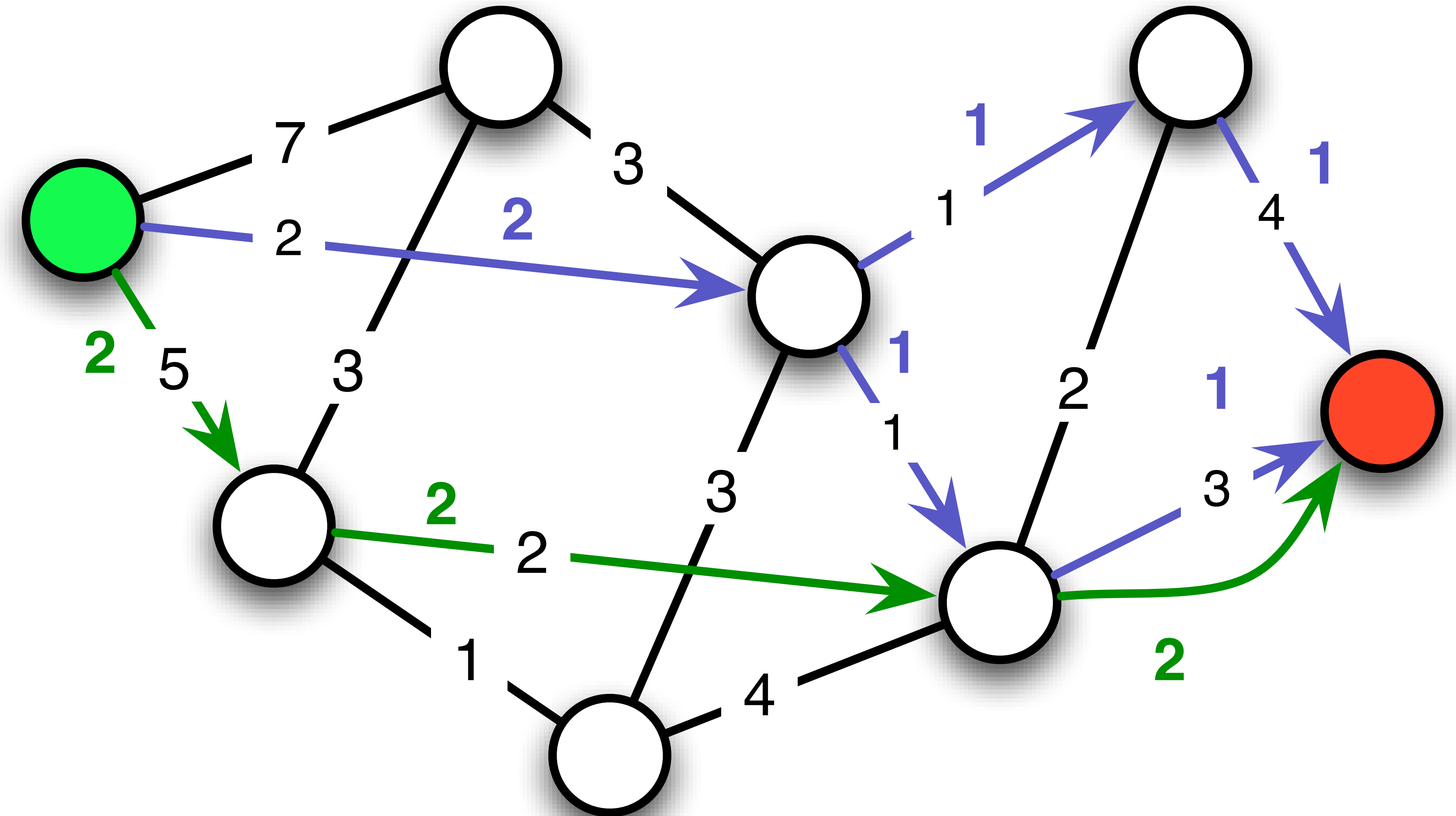

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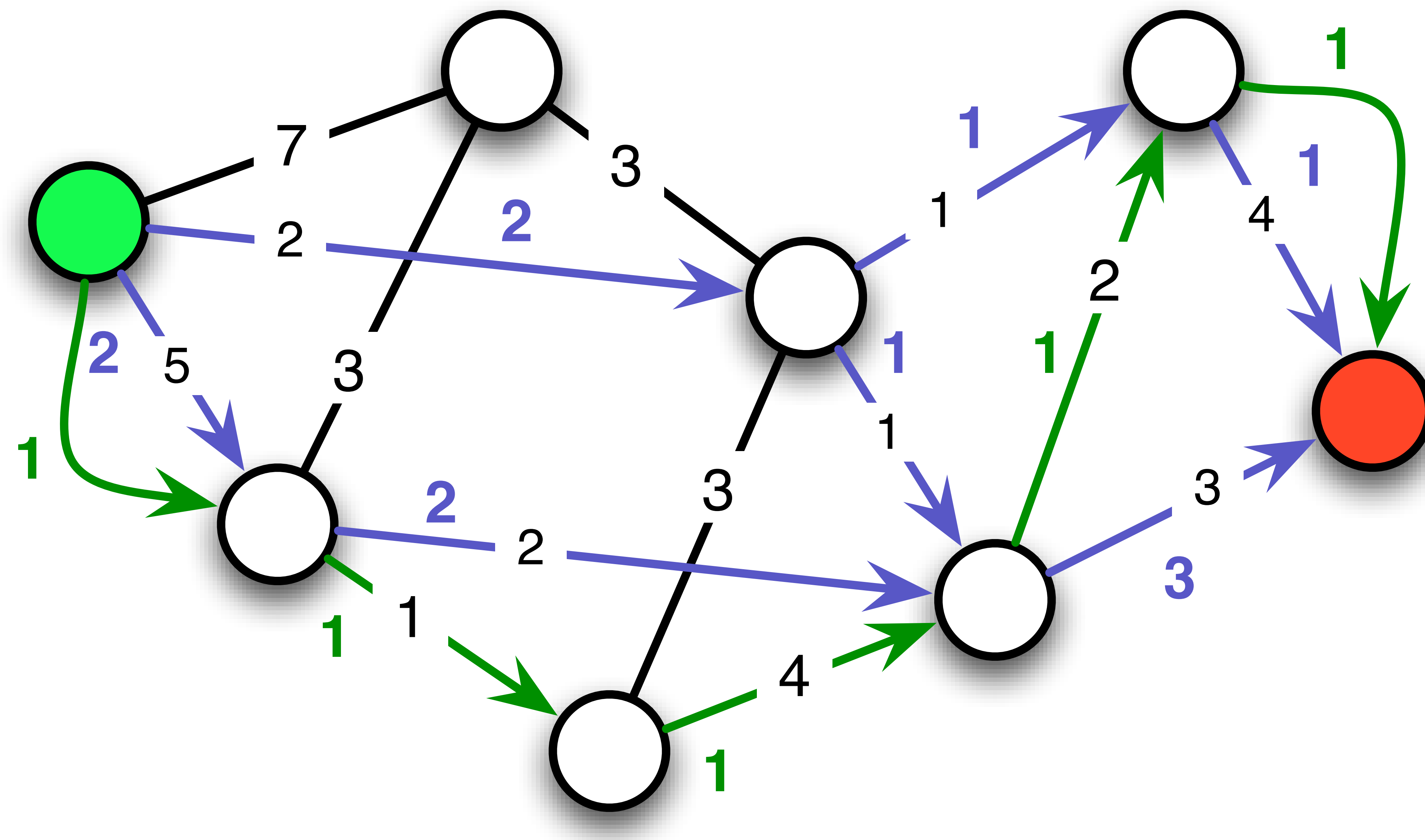

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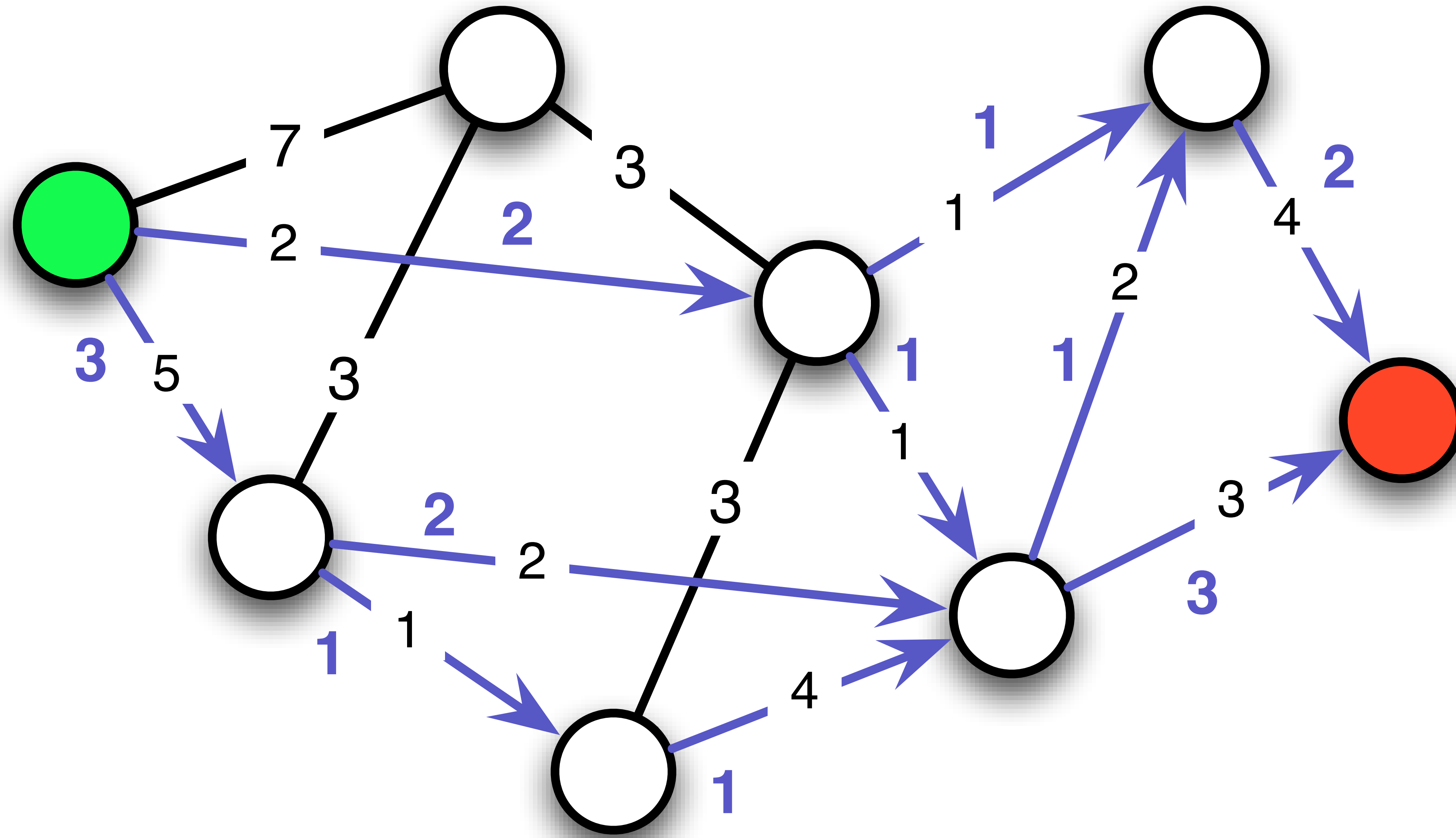

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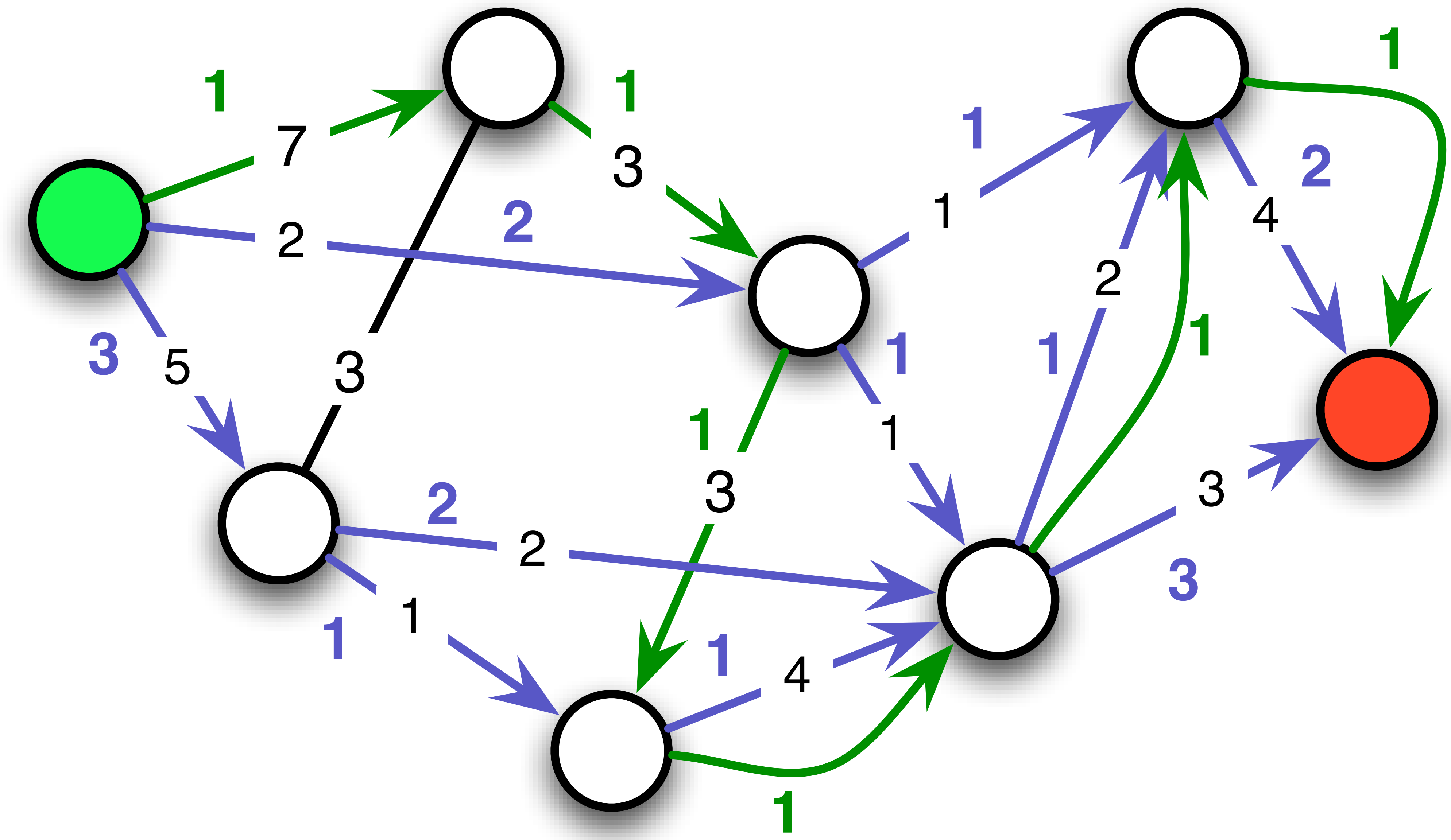

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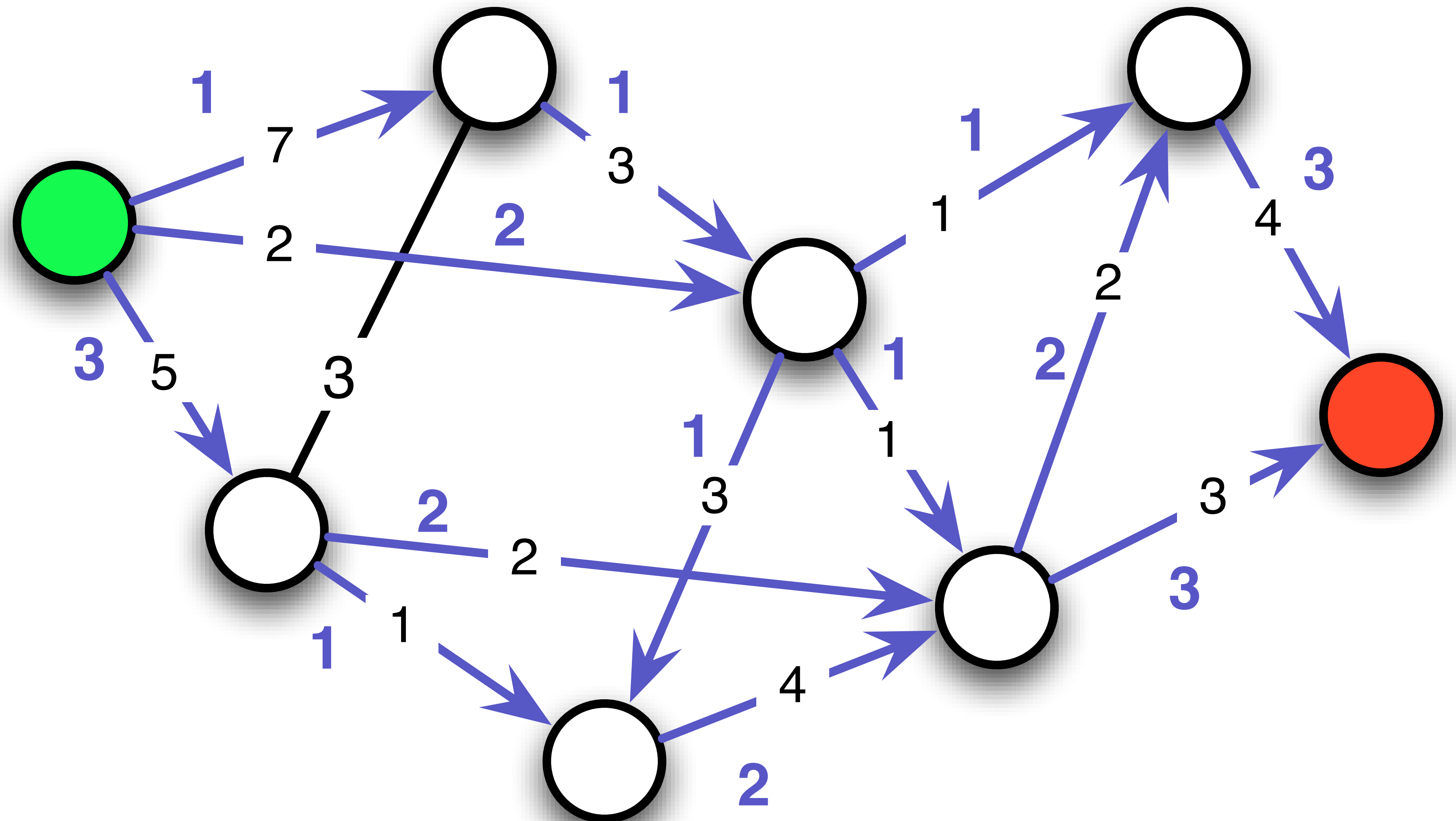
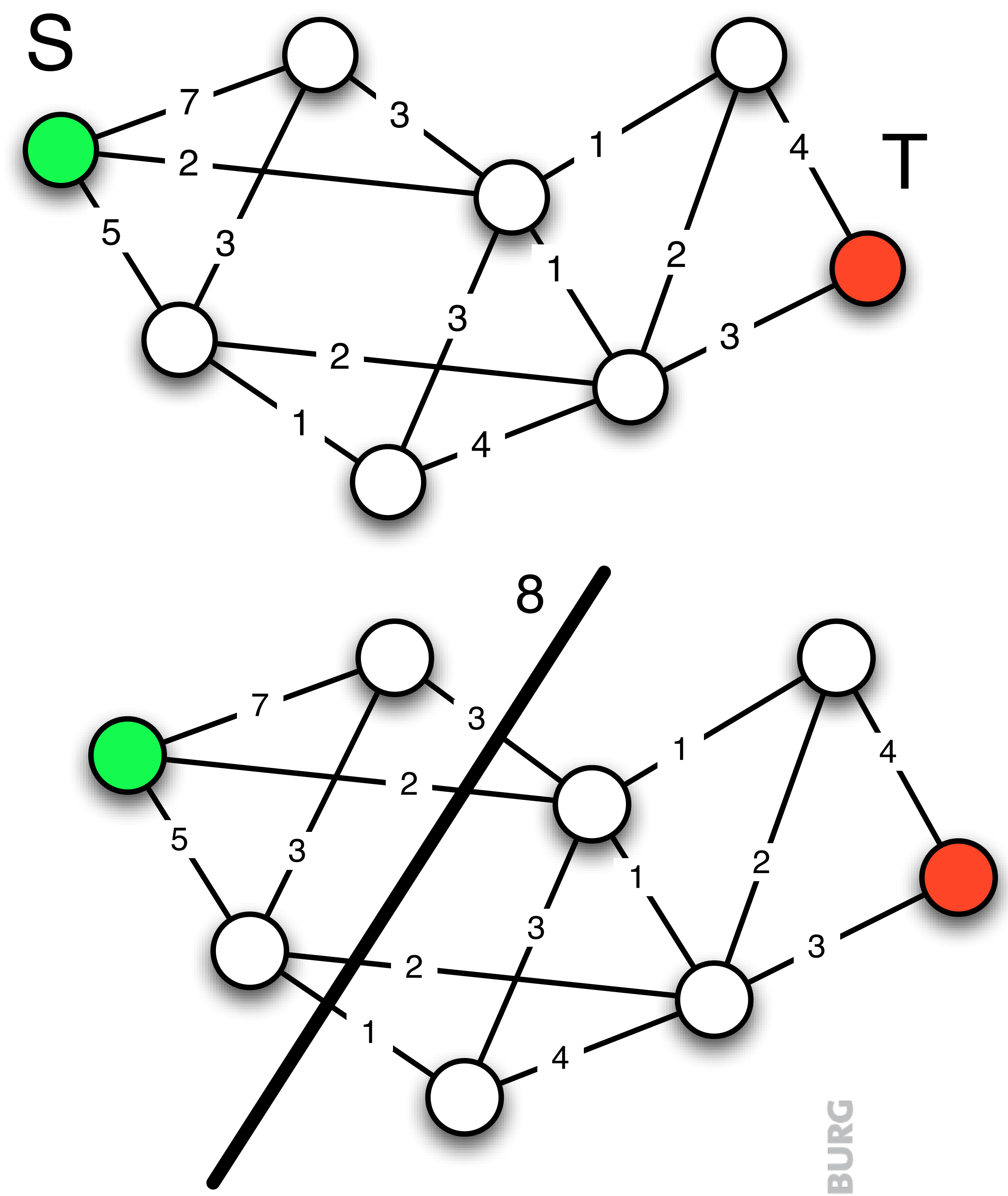


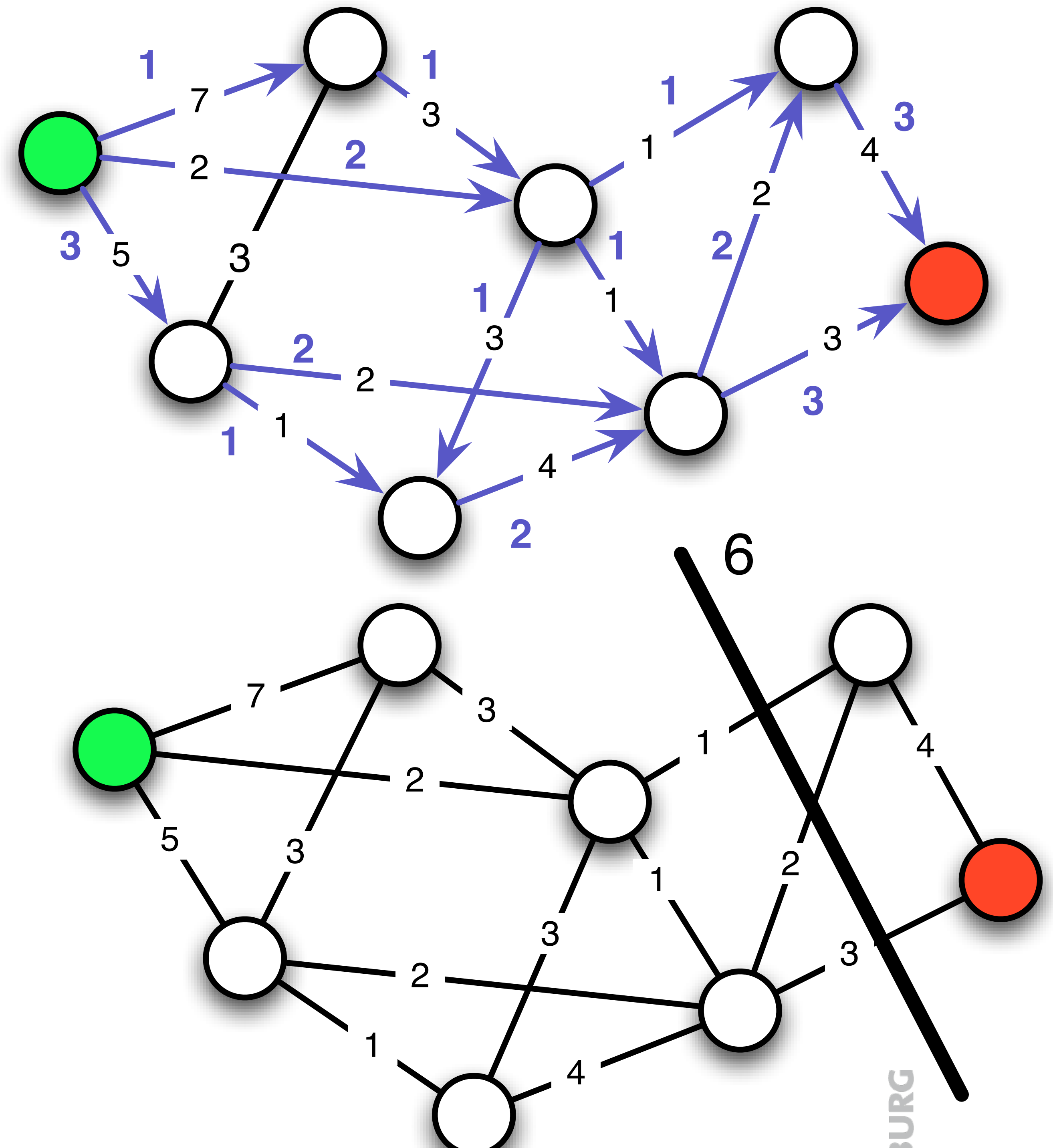
Minimum Cut in Networks

- Motivation
  - Find bottleneck in networks
- Definition
  - Min Cut problem
  - Given
    - graph G=(V,E)
    - capacity function w: E → R+0,
    - sources S and targets T
  - Find minimum cut between S and T
- A cut C is a set of edges
  - such that every path from a node of S to a node of T, contains an edge of C
- The size of a cut is $\sum_{e \in C} w(e)$



Min-Cut-Max-Flow Theorem

- Theorem
  - The minimum cut equals the maximum flow
- Algorithms for minimum cut
  - can be obtained from the maximum flow algorithms



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