Peer-to-Peer Networks
07 Degree Optimal Networks

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Degree Optimal Networks

Koorde
M. Frans Kaashoek and David R. Karger 2003
De Brujin Graph

out degree: 2
in degree: 2

diameter: 2

# nodes: $n = 2^m$

$= \log_2 m$
Consider binary string $s$ of length $m$

- shuffle operation:
  \[
  \text{shuffle}(s_1, s_2, s_3, \ldots, s_m) = (s_2, s_3, \ldots, s_m, s_1)
  \]

- exchange:
  \[
  \text{exchange}(s_1, s_2, s_3, \ldots, s_m) = (s_1, s_2, s_3, \ldots, \neg s_m)
  \]

- shuffle exchange:
  \[
  \text{SE}(S) = \text{exchange}(\text{shuffle}(S)) = (s_2, s_3, \ldots, s_m, \neg s_1)
  \]

Observation:
Every string $a$ can be transformed into a string $b$ by at most $m$ shuffle and shuffle exchange operations.
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Every string $a$ can be transformed into a string $b$ by at most $m$ shuffle and shuffle exchange operations. Example:

From

$$\begin{array}{ccccccc}
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}$$

to

$$\begin{array}{ccccccc}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}$$

via

$$\text{SE SE SE S SE S S operations}$$
1011
0001
1010
S E S E S

St. 0001100000
1011
0001000000
S S S E S E S S S S
A De Bruijn graph consists of $n=2^m$ nodes, each representing an $m$ digit binary strings.

Every node has two outgoing edges:
- $(u, \text{shuffle}(u))$
- $(u, \text{SE}(u))$

**Lemma**
- The De Bruijn graph has degree 2 and diameter $\log n$.

**Koorde = Ring + DeBruijn-Graph**
Koorde = Ring + DeBruijn-Graph

- Consider ring with $2^m$ nodes and De Bruijn edges
**Note**

- shuffle(s₁, s₂, ..., sₘ) = (s₂, ..., sₘ, s₁)
  - shuffle (x) = (x div 2^{m-1})+(2x) mod 2^{m}
- SE(S) = (s₂, s₃, ..., sₘ, ¬ s₁)
  - SE(x) = 1-(x div 2^{m-1})+(2x) mod 2^{m}
- Hence: Then neighbors of x are
  - 2x mod 2^{m} and
  - 2x+1 mod 2^{m}
Virtual DeBruijn Nodes

- To avoid collisions we choose
  - $m > (2+c) \log(n)$
- Then the probability of two peers colliding is at most $n^{-c}$
- But then we have much more nodes in the graph than peers in the network
- Solution
  - Every peer manages all DeBruijn nodes between his position and his successor on the ring
  - only for incoming edges
  - outgoing edges are considered only from the peer's position on the ring
Properties of Koorde

- **Theorem**
  - Every node has four pointers
  - Every node has at most $O(\log n)$ incoming pointers w.h.p.
  - The diameter is $O(\log n)$ w.h.p.
  - Lookup can be performed in time $O(\log n)$ w.h.p.

- **But:**
  - Connectivity of the network is very low.
Properties of Koorde

- Theorem
  - 1. Every node has four pointers
  - 2. Every node has at most $O(\log n)$ incoming pointers w.h.p.

- Proof:
  - 1. follows from the definition of the De Bruijn graph and the observation that only non-virtual nodes have outgoing edges
  - 2. The distance between two peers is at most $c(\log n)/n^2$ with high probability
  - The number of nodes pointing to this distance is therefore at most $c(\log n)$ with high probability
Properties of Koorde

- Theorem
  - The diameter is $O(\log n)$ w.h.p.
  - Lookup can be performed in time $O(\log n)$ w.h.p.

- Proof sketch:
  - The minimal distance of two peers is at least $n^{-c} 2^{m}$ w.h.p.
  - Therefore use only the $c \log n$ most significant bits in the routing
    - since the prefix guarantees that one end in the responsibility area of a peer
  - Follow the routing algorithm on the De Bruijn-graph until one ends in the responsibility area of a peer
Consider alphabet using \( k \) letters, e.g. \( k = 3 \)

Now, every \( k \)-De Bruijn-node has successors
- \( (kx \mod km) \)
- \( (kx + 1 \mod km) \)
- \( (kx + 2 \mod km) \)
- ... \( (kx + k - 1 \mod km) \)

Diameter is reduced to
- \( \frac{\log m}{\log k} \)

Graph connectivity is increased to \( k \)
k-Koorde

- Straight-forward generalization of Koorde
  - by using k-De Bruijn graphs
- Improves lookup time and messages to $O((\log n)/(\log k))$ steps
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