Wireless Sensor Networks
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Options for topology control

Control node activity – deliberately turn on/off nodes

Control link activity – deliberately use/not use certain links

Flat network – all nodes have essentially same role

Hierarchical network – assign different roles to nodes; exploit that to control node/link activity

Power control

Backbones

Clustering
Geometric Spanners with Applications in Wireless Networks

1. Introduction
   - Definition of Geometric Spanners
   - Motivation
   - Related Work
2. Spanners versus Weak Spanners
3. Spanners versus Power Spanners
4. Weak Spanners versus Power Spanners
   - Weak Spanners are Power Spanners if Exponent > Dimension
   - Weak Spanners are Power Spanners if Exponent = Dimension
   - Weak Spanners are not always Power Spanners if Exponent < Dimension
   - Fractal Dimensions
5. Applications in Wireless Networks
6. Conclusions
Geometric Spanner Graphs

A Graph $G = (V, E)$ with $V \subseteq \mathbb{R}$ where for all $u, v \in V$ there exists a path $P = (u = u_1, u_2, \ldots, u_\ell = v)$ with limited length:

$||P|| := \sum_{i=2}^{\ell} |u_i - u_{i-1}| \leq c \cdot |u - v|$  

-in a limited radius:

$\max_{i=1,\ldots,\ell} |u - u_i| \leq c \cdot |u - v|$  

limited energy costs:

$||P||^\delta := \sum_{i=2}^{\ell} |u_i - u_{i-1}|^\delta \leq c \cdot |u - v|^\delta$

- $c$-Spanner Graph
- weak $c$-Spanner Graph
- $(c, \delta)$-Power-Spanner Graph
Geometric Spanners with Applications in Wireless Networks

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6. Conclusions
Spanners versus Weak Spanners

- **Fact**
  - Every c-Spanner is also a c-Weak Spanner

- **Theorem**
  - There are Weak Spanner which are no Spanners

- **Proof Idea [Eppstein]**: use fractal construction
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Spanners versus Power Spanners

- **Theorem**
  - For $\delta > 1$, every $c$-Spanner is also a $(c^\delta, \delta)$-Power Spanner

- **Proof:**
  - Consider a path $P$ in the $c$-spanner with stretch factor $c$
  - This path is already a $\delta$-Power Spanner graph with stretch factor $c^\delta$:

\[
\|P\|^{\delta} = \sum_{i=1}^{\ell-1} \|P_i\|^{\delta} \leq \sum_{i=1}^{\ell-1} (c \cdot |u_i - u_{i+1}|)^{\delta}
\]

\[
= c^\delta \cdot \sum_{i=1}^{\ell-1} (|u_i - u_{i+1}|)^{\delta} = c^\delta \cdot \|P_{OPT}\|^{\delta}
\]
(Weak) Spanners versus Power Spanners

➢ Theorem
  – For $\delta > 1$ there is a graph family of $(c, \delta)$-Power Spanners which are no weak C-Spanners for any constant C.

➢ Proof:
  – ...

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Weak Spanners are Power Spanners if Exponent > Dimension (I)

Lemma
- Let G be a weak c-spanner. Then, there is a path from nodes u to v in this graph G which as a subgraph of G is a weak 2c-spanner.

Proof sketch
- Wlog. let |u-v| = 1
- Start with a weak c-spanner path from u to v
- If two nodes x,y in this path are closer than 1/2
  - and the interior points of the sub-path (x,...,y) are outside the disk with center x and radius c/2
  - then construct the weak c-spanner path P’ from x to y
  - and substitute the sub-path (x,...,y) with this sub-path P’
- Repeat this process for nodes with distance 1/4, 1/8,...
- The new path is then within a circle of radius 2c with center u
Weak Spanners are Power Spanners if Exponent > Dimension (II)

Lemma

Let \( P = (u_1, \ldots, u_\ell) \) be a weak 2c-spanner, \( u_i \in \mathbb{R}^D \), \( |u_1 - u_\ell| = 1 \). Then \( P \) contains at most \((8c + 1)^D\) edges of length greater than \( c \); more generally, \( P \) contains at most \((8c + 1)^D(2^D)^k\) edges of length greater than \( c/2^k \).

Proof sketch

- Consider the D-dimensional spheres of radius \( c \)
- If the path is a weak spanner then for all pairs \( u, v \) of the path nodes the weak spanner property is valid, hence \( |u, v| > 1/2 \)
- How many nodes can be gathered in the sphere of radius 1/2?
- Consider spheres of radius 1/4. These spheres do not intersect.
- For an upper bound divide the volume of the radius \( c \)-sphere by the volumes of the small spheres

Analogous for shorter edges
Weak Spanners are Power Spanners if Exponent > Dimension (III)

Theorem

- For $\delta > D$ let $G = (V, E)$ be a weak $c$-spanner with $V \subset \mathbb{R}^D$. Then $G$ is a $(C, \delta)$-power spanner for

$$C' := (8c + 1)^D \cdot \frac{(2c)^\delta}{1 - 2^{D-\delta}}$$

Proof sketch:

- Choose edge lengths from $[2^i, 2^{i+1}]$
- Sum over the edge lengths up to length $c |u,v|$ and use:

more generally, $P$ contains at most $(8c + 1)^D (2^D)^k$ edges of length greater than $c/2^k$.

- This leads to a converging sum for $\delta < D$
Weak Spanners are not Power Spanners if Exponent < Dimension

Theorem

To any $\delta < D$, there exists a family of geometric graphs $G = (V, E)$ with $V \subset \mathbb{R}^D$ which

- are weak $c$-spanners for a constant $c$
- but not $(C, \delta)$-power spanners for any fixed $C$. 

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Weak Spanners are Power Spanners if Exponent = Dimension

Theorem
- Let \( G = (V , E) \) be a weak \( c \)-spanner with \( V \subset \mathbb{R}^D \). Then \( G \) is a \((C, D)\)-power spanner for \( C := O(c^{4D}) \).

Proof strategy:
1. For a two nodes \((u,v)\) with \(|u,v|=1\)
   - Consider a bounding square of side length \(4c\)
2. For \(k = 0,1,2,..\)
   - Consider edges of lengths \([c^{\beta^{-k-1}},c^{\beta^{-k}}]\) for some constant \(\beta > 1\)
3. In each iteration use “clean-up” to produce empty space
4. If long edges exist, then at least one empty square of volume \(\Omega(\beta^{-Dk})\) exist.

\[
\sum_{\text{edges in round } k} \left( \frac{\text{edge lengths in round } k}{\text{in round } k} \right)^D = O \left( \frac{\text{Volume of empty space added in round } k}{\text{volume added in round } k} \right)
\]
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Conclusions

- Complete characterization of the relationships of Spanners, Weak Spanners and Power Spanners

<table>
<thead>
<tr>
<th></th>
<th>( c )</th>
<th>( c )</th>
<th>( (c^\delta, \delta) )</th>
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</thead>
<tbody>
<tr>
<td><strong>( c )-spanner</strong></td>
<td>(unbounded)</td>
<td>( c )</td>
<td>( (O(c^{2D+\epsilon}/(1 - 2^{-\epsilon})), D + \epsilon) )</td>
</tr>
<tr>
<td><strong>weak ( c )-spanner</strong></td>
<td>(unbounded)</td>
<td>( c )</td>
<td>( (O(c^{4D}), D) )</td>
</tr>
<tr>
<td>( (c, \delta) )-power spanner</td>
<td>(unbounded)</td>
<td>(unbounded)</td>
<td>for ( \Delta &gt; \delta ): ( (c^{\Delta/\delta}, \Delta) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>for ( \Delta &lt; \delta ): (unbounded, ( \Delta ))</td>
</tr>
</tbody>
</table>

**is a**

- spanner
- weak spanner
- power spanner
Delaunay Graph

➤ Definition
- Triangularization of a point set p such that no point is inside the circumcircle of any triangle

➤ Facts
- Dual graph of the Voronoi-diagram
- In 2-D
  • edge flipping leads to the Delaunay-graph
    ▪ Flip edge if circumcircle condition is not fulfilled
  • planar graphs
  • 5.08-spanner graph
- Problem: Might produce very long links
Yao-Graph

- Choose nearest neighbor in each sector
- \textit{c-spanner},
  - with stretch factor \[1/(1 - 2\sin(\theta/2))\]
  - Simple distributed construction
  - High (in-) degree

\[YG_6\]
Thank you