Algorithm Theory 2008/09
Proposel for a solution of exercise sheet 3
Nov 12, 2008

**TASK 1**

1. Specify all primitive 8-th roots of unity.

From the cancellation lemma follows that \{\omega_0^8, \omega_2^8, \omega_4^8, \omega_6^8\} equals the 4-th roots of unity. Therefore they are not primitive 8-th roots of unity. The other roots \{\omega_1^8, \omega_3^8, \omega_5^8, \omega_7^8\} are primitive as the following table shows.

<table>
<thead>
<tr>
<th>x</th>
<th>(x^2)</th>
<th>(x^4)</th>
<th>(x^3)</th>
<th>(x^6)</th>
<th>(x^1)</th>
<th>(x^8)</th>
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</thead>
<tbody>
<tr>
<td>(\omega_1^8)</td>
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<td>(\omega_4^8)</td>
<td>(\omega_5^8)</td>
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</table>

2. Calculate the product of \(p(x) = 3x - 1\) and \(q(x) = 2x + 5\).

(a) Compute the FFT of \(p(x)\) and \(q(x)\).

```
DFT_4(p)
FFT([-1,3], 2) for n = 4
\(d[0] = \text{FFT}(1, 1) = -1\)
\(d[1] = \text{FFT}(3, 1) = 3\)
\(\omega = 1, k = 0\)
\(d_0 = d[0] + \omega d[1] = -1 + 3 = 2\)
\(d_2 = d[0] - \omega d[1] = -1 - 3 = -4\)
\(\omega = e^{2\pi i/4} = i, k = 1\)
\(d_1 = d[0] + \omega d[1] = -1 + 3i\)
\(d_3 = d[0] - \omega d[1] = -1 - 3i\)
```
We calculate $DFT_4$, because $pq$ is of degree 4.

(b) Give the point-value representation of $pq$ at the $k$-th roots of unity for an appropriate choice of $k$.

$\begin{array}{cccc}
\text{i} & \omega_i & p(\omega_i) & q(\omega_i) \\
0 & 1 & 2 & 7 \\
1 & i & -1 + 3i & 5 + 2i & -11 + 13i \\
2 & -1 & -4 & 3 & -12 \\
3 & -i & -1 - 3i & 5 - 2i & -11 - 13i \\
\end{array}$

Point value representation:
$\{(1, 14), (i, -11 + 13i), (-1, -12), (-i, -11 - 13i)\}$

(c) Compute the interpolation by using the FFT algorithm

$$a = \frac{1}{n} \left( r(\omega_4^0), r(\omega_4^1), r(\omega_4^2), r(\omega_4^3) \right)$$
$$= \frac{1}{n} \left( r(1), r(\omega_4^0), r(\omega_4^2), r(\omega_4^1) \right)$$
$$y = [pq(\omega_4^0), pq(\omega_4^1), pq(\omega_4^2), pq(\omega_4^3)] = [14, -11 - 13i, -12, -11 + 13i]$$

Note, that we have to re-order the input for the FFT.
\[\begin{align*}
d^{[0]} &= FFT([14, -12], 2) \\
&= ( FFT([14], 1) + FFT([-12], 1), FFT([14], 1) - FFT([-12], 1) ) \\
&= (2, 26) \\
d^{[1]} &= FFT([-11 - 13i, -11 + 13i], 2) \\
&= ( FFT([-11 - 13i], 1) + FFT([-11 + 13i], 1), ...) \\
&= (-22, -26i)
\end{align*}\]

\[\begin{align*}
\omega &= 1, \ k = 0 \\
d_0 &= d_0^{[0]} + \omega d_0^{[1]} = 2 + 1 \cdot (-22) = -20 \\
d_2 &= d_0^{[0]} - \omega d_0^{[1]} = 2 - 1 \cdot (-22) = 24 \\
\omega &= e^{2\pi i/4} \cdot 1 = i, \ k = 1 \\
d_1 &= d_1^{[0]} + \omega d_1^{[1]} = 26 + i \cdot (-26i) = 52 \\
d_3 &= d_1^{[0]} - \omega d_1^{[1]} = 26 - i \cdot (-26i) = 0
\end{align*}\]

\[\begin{align*}
DFT_4(r) &= (-20, 52, 24, 0) \\
a &= \frac{1}{4}DFT_4(r) = (-5, 13, 6, 0) \\
pq(x) &= -5 + 13x + 6x^2 + 0x^3
\end{align*}\]

(d) Check the correctness of the result.

\[pq(x) = (3x - 1)(2x + 5) = 6x^2 + 13x - 5\]