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Algorithm Theory

7 Hashing

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Overview

- **Introduction**
- **Universal Hashing**
- **Perfect Hashing**

The Dictionary Problem

Given: Universe $U = [0 \ldots N\text{-}1]$, where N is a natural number.

Goal: Maintain a set $S \subseteq U$ under the following operations

- **Search(x,S):** Is $x \in S$?
- **Insert(x,S):** Insert x into S if not already in S.
- **Delete(x,S):** Delete x from S

Naive Implementation

Array A[0…*N*-1] **where** **A[*i*] = 1 ⇔ *i* ∈ *S***

Each operation takes constant (O(1)) time but required memory space is $\Theta(N)$.

A

0	0	1	1	0	1	1	1	0	0	1

0 … *N*-1

Goal: Space O(|*S*|) and expected time O(1).

Idea of Hashing

Use an Array of length $O(|S|)$.

Compute the position, where to store an element using a function defined on the keys

Universe	$U = [0 \ldots N\text{-}1]$
Hash table	Array $T[0 \ldots m\text{-}1]$
Hash function	$h: U \rightarrow [0 \ldots m\text{-}1]$

Element $x \in S$ is stored in $T[h(x)]$

Example

$N = 100$; $U = [0 \ldots 99]$; $m = 7$; $h(x) = x \bmod 7$; $S = \{3, 19, 22\}$

0	
1	22
2	
3	3
4	
5	19
6	

If 17 is inserted next, a collision arises because $h(17) = 3$.

Possible Collision Resolutions

- Hashing with chaining: $T[i]$ contains a list of elements

- Hashing with open addressing: Instead of one address for an element there are m many that are probed sequentially

- Universal Hashing: Choose a hash function such that only few collisions occur. Collisions are resolved by a chained lists.

- Perfect hashing: Choose a hash function such that no collisions occur.

Universal Hashing

Idea: Use a class H of hash functions. The hash function $h \in H$ actually used is chosen uniformly at random from H.

Goal: For each $S \subseteq U$, the expected time of each operation is $O(1 + \beta)$ where $\beta = |S|/m$ is the load factor of the table.

Property of H: For two arbitrary elements $x,y \in U$, only few $h \in H$ lead to a collision ($h(x) = h(y)$).

Universal Hashing

Definition: Let N and m natural numbers. A class

$H \subseteq \{ h : [0\ldots N\text{-}1] \rightarrow [0\ldots m\text{-}1] \}$ is universal if for all

$x,y \in U = [0\ldots N\text{-}1]$, $x \neq y$:

$$\frac{|\{h \in H : h(x) = h(y)\}|}{|H|} \leq \frac{1}{m}$$

Intuition: An h chosen uniformly at random is as good as if the table position of the elements are chosen uniformly at random.

A Universal Class of Functions

Let N, m be natural numbers, N is prime.

For numbers $a \in \{1, \ldots, N\text{-}1\}$ and $b \in \{0, \ldots, N\text{-}1\}$, let

$h_{a,b} : U = [0 \ldots N\text{-}1] \rightarrow \{0, \ldots, m\text{-}1\}$ be defined as:

$$h_{a,b}(x) = ((ax + b) \bmod N) \bmod m$$

Theorem: $H = \{h_{a,b}(x) \mid 1 \leq a < N \text{ und } 0 \leq b < N\}$ is a universal class of hash functions

Proof

Consider a fixed pair x,y with $x \neq y$.

$h_{a,b}(x) = ((ax+b) \bmod N) \bmod m \quad h_{a,b}(y) = ((ay+b) \bmod N) \bmod m$

1. Pairs (q,r) with $q = (ax+b) \bmod N$ and $r = (ay+b) \bmod N$

   for variable a,b take the whole range $0 \leq q,r < N$ with $q \neq r$

   - $x \neq y$ implies $q \neq r$, because $q = r$ implies $a(x-y) = 0 \bmod N$

   - different a,b yield different pairs (q,r).

     $(ax+b) \bmod N = q \qquad (ay+b) \bmod N = r$

     $(a'x+b') \bmod N = q \qquad (a'y+b') \bmod N = r$

     imply $(a-a')(x-y) = 0 \bmod N$

Proof

Consider a fixed pair x,y with $x \neq y$.

$h_{a,b}(x) = ((ax+b) \bmod N) \bmod m$ $h_{a,b}(y) = ((ay+b) \bmod N) \bmod m$

2. How many pairs (q,r) with $q = (ax+b) \bmod N$ and $r = (ay+b) \bmod N$ are mapped into the same residue class mod m?

   For a fixed q, there are only $(N\text{-}1)/m$ numbers r with $q \bmod m = r \bmod m$ and $q \neq r$.

   $$|\{h \in H : h(x) = h(y)\}| \leq \underbrace{N}_{\text{\# possible values for q}} \underbrace{(N-1)/m}_{\text{\# possible values for r leading to a collision}} = |H|/m$$

Analysis of the Operations

Assumptions:

1. *h* is chosen uniformly at random from a universal class *H*
2. Collisions are resolved by chained lists

For $h \in H$ and $x,y \in U$ let

$$\delta_h(x,y) = \begin{cases} 1 & h(x) = h(y) \text{ and } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

$\delta_h(x,S) = \sum_{y \in S} \delta_h(x,y)$ is the number of elements in T[h(x)] different from x when S is stored.

Analysis of the Operations

Theorem: Let H be a universal class and $S \subseteq U = [0\ldots N\text{-}1]$ with $|S| = n$.

1. For any $x \in U$:

$$\underbrace{\frac{1}{|H|}\sum_{h \in H}(1 + \delta_h(x,S))}_{} \leq \begin{cases} 1 + n/m & x \notin S \\ 1 + (n-1)/m & x \in S \end{cases}$$

exp. number of collisions with x $\longrightarrow$ exp. chain length at position x $\longrightarrow$ exp. number of operations for accessing x

2. The expected time of the operations Search, Insert, and Delete is $O(1 + \beta)$, where $\beta = n/m$ is the load factor.

Proof

1.
$$\begin{aligned}
\sum_{x \in H} (1 + \delta_h(x,S)) &= |H| + \sum_{h \in H} \sum_{y \in S} \delta_h(x,y) \\
&= |H| + \sum_{y \in S} \sum_{h \in H} \delta_h(x,y) \\
&= |H| + \sum_{y \in S \setminus \{x\}} \sum_{h \in H} \delta_h(x,y) \\
&\leq |H| + \sum_{y \in S \setminus \{x\}} \frac{|H|}{m} \\
&\leq \begin{cases} |H| \ (1 + \frac{n}{m}) & x \notin S \\ |H| \ (1 + \frac{n-1}{m}) & x \in S \end{cases}
\end{aligned}$$

2. follows from 1.

Perfect Hashing

Choose a hash function that is injective (i.e. one-to-one) on the set S to be stored. (Assumption: S is known in advance.)

Two-level hashing scheme

1. In the first level, S is partitioned into "short lists".
   (hashing with chaining)
2. In the second level for each list, a separate injective hash function used.

Constructions of Injective Hash Functions

Let $U = [0 \ldots N\text{-}1]$

For $k \in \{1,\ldots,N\text{-}1\}$, let

$$h_k : U \rightarrow \{0,\ldots,m\text{-}1\}$$

$$x \rightarrow ((kx) \bmod N) \bmod m$$

Let $S \subseteq U$. Is it possible to choose k such that h_k restricted to S is injective?

h_k restricted to S is injective if for all $x,y \in S$, $x \neq y$,

$$h_k(x) \neq h_k(y)$$

A Measure for the Violation of Injectivity

For $0 \leq i \leq m\text{-}1$ and $1 \leq k \leq N\text{-}1$

$$b_{ik} = |\{ x \in S : h_k(x) = i \}|$$

Then:

$$|\{ (x,y) \in S^2 : x \neq y \text{ and } h_k(x) = h_k(y) = i \}| = b_{ik} (b_{ik} - 1)$$

Define

$$B_k = \sum_{i=0}^{m-1} b_{ik}(b_{ik} - 1)$$

B_k measure to which extent h_k restricted to S is not injective.

Injectivity

Lemma 1: h_k restricted to S is injective $\Leftrightarrow$ $B_k < 2$

Proof:

$B_k < 2 \quad \Rightarrow \quad B_k \leq 1 \quad \Rightarrow \quad b_{ik}\,(b_{ik} - 1) \in \{0,1\}$ for all i

$\Rightarrow \quad b_{ik} \in \{0,1\} \quad \Rightarrow \quad h_k$ restricted to S is injective

h_k restricted to S is injective $\quad \Rightarrow \quad b_{ik} \in \{0,1\}$ for all i

$\Rightarrow \quad B_k = 0$

Injectivity

Lemma 2: Let N be a prime number, $S \subseteq U = [0\ldots N\text{-}1]$ with $|S| = n$. Then

$$\sum_{k=1}^{N-1} B_k \leq 2\frac{n(n-1)}{m}(N-1)$$

If $m > n(n\text{-}1)$, then there exists B_k with $B_k < 2$,

i.e. there is an h_k, that is injective on S.

Proof of Lemma 2

$$\sum_{k=1}^{N-1} B_k = \sum_{k=1}^{N-1} \sum_{i=0}^{m-1} b_{ik}(b_{ik} - 1)$$

$$= \sum_{k=1}^{N-1} \sum_{i=0}^{m-1} |\{(x,y) \in S^2 \ : \ x \neq y, \ h_k(x) = h_k(y) = i\}$$

$$= \sum_{\substack{(x,y) \in S^2 \\ x \neq y}} |\{k \ : \ h_k(x) = h_k(y) \qquad \}|$$

Let $(x,y) \in S^2$, $x \neq y$, be fixed. How many k exist with $h_k(x) = h_k(y)$?

Proof of Lemma 2

$$h_k(x) = h_k(y)$$

$$\Leftrightarrow ((kx) \bmod N) \bmod m = ((ky) \bmod N) \bmod m$$

$$\Leftrightarrow (kx \bmod N - ky \bmod N) \bmod m = 0$$

$$\Leftrightarrow k(x - y) \bmod N = cm$$

$q = k(x-y) \bmod N$

-- *different* k, k' result in different q, q'.

$k(x-y) \bmod N = q$ $\quad\quad\quad$ $k'(x-y) \bmod N = q$

$(k-k')(x-y) = c'N$

-- only $\lceil (N-1)/m \rceil$ many q are mapped into the same residue class mod m

Implications

Corollary 1: There are at least $(N\text{-}1)/2$ many k with $B_k \leq 4n(n\text{-}1)/m$.

Such a k can be found in expected time $O(m+n)$.

Proof: Suppose that there are less than $(N\text{-}1)/2$ many k with $B_k \leq 4n(n\text{-}1)/m$.

Then there at least $(N\text{-}1)/2$ many k with $B_k > 4n(n\text{-}1)/m$

$$\Rightarrow \sum_{k=1}^{N-1} B_k > \frac{N-1}{2} \frac{4n(n-1)}{m} = \frac{N-1}{m} 2n(n-1)$$

With probability at least 1/2, a k chosen at random fulfills the condition. The expected number of trials is at most 2.

Implications

Corollary 2:

a) Let $m = 2n(n-1)+1$. Then at least $(N-1)/2$ of the h_k are injective on S. Such an h_k can be found in expected time $O(m+n)=O(n^2)$.

b) Let $m = n$. Then for at least $(N-1)/2$ of the h_k it holds that $B_k \leq 4(n-1)$. Such an h_k can be found in expected time $O(n)$.

Two-Level Scheme

$S \subseteq U = [0 \ldots N\text{-}1]$ $\quad\quad |S| = n = m$

Idea: Use Corollary 2b and divide S into subsets of size at most $O(\sqrt{n})$.

Use Corollary 2a for each subset.

1. Choose k with $B_k \leq 4(n\text{-}1) \leq 4n$.

   $h_k : x \rightarrow ((kx) \bmod N) \bmod n$

2. $W_i = \{ x \in S : h_k(x) = i \}$, $\quad b_i = |W_i|$, $\quad m_i = 2b_i (b_i - 1)+1$ for $0 \leq i \leq n\text{-}1$

   Choose k_i such that

$$h_{k_i} : x \rightarrow (k_i x \bmod N) \bmod m_i$$

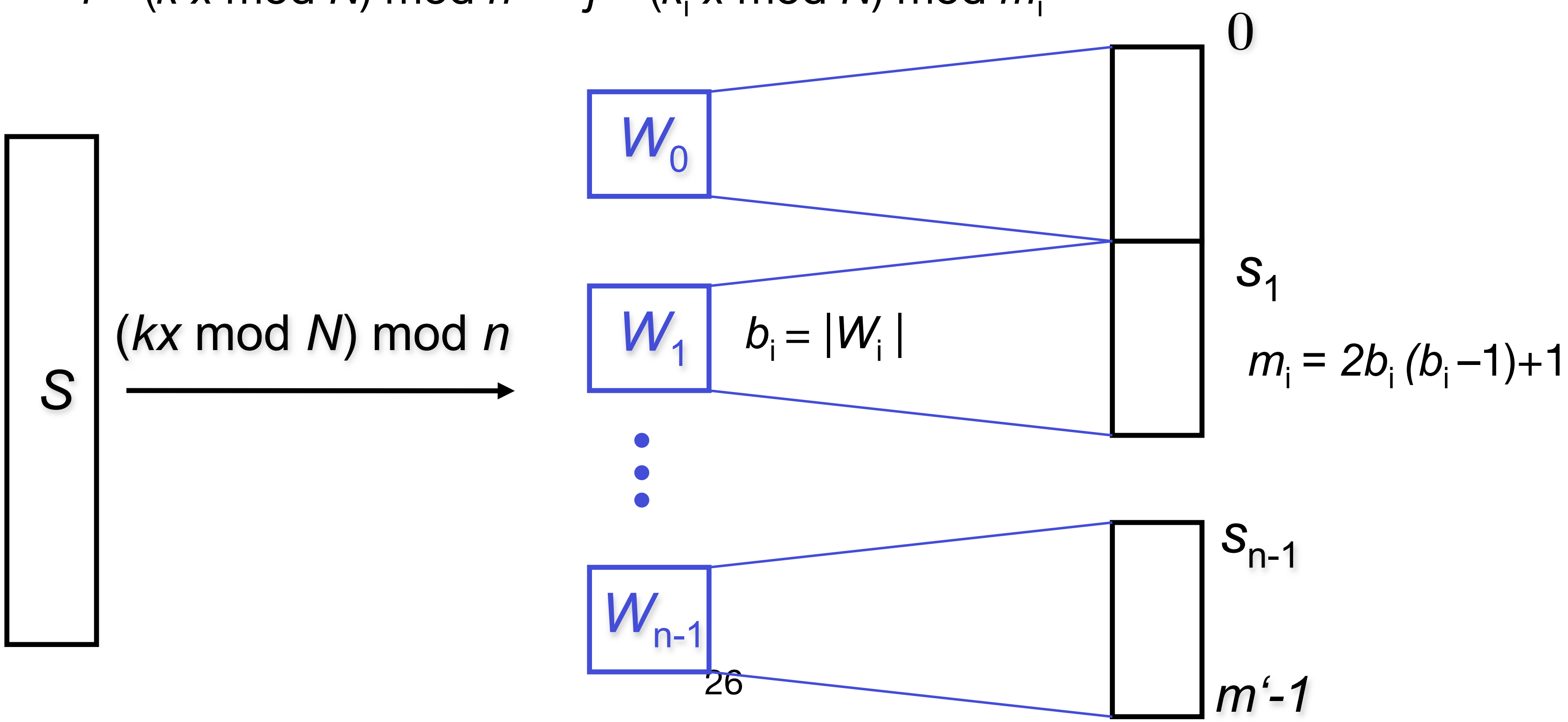
restricted to W_i **is injective**.

Two-Level Scheme

3. $S_i = \sum_{j<i} m_j$

Store $x \in S$ in table positions $T[s_i + j]$ where

$i = (k\,x \bmod N) \bmod n$ $\quad j = (k_i\,x \bmod N) \bmod m_i$



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Space Required for Hash Table and Functions

$$
\begin{aligned}
m &= \sum_{i=0}^{n-1} m_i = \sum_{i=0}^{n-1} (2b_i(b_i - 1) + 1) = n + 2B_k \\
&\leq n + 8(n-1) \;\leq\; 9n
\end{aligned}
$$

Additional space is required for storing k_i, m_i and s_i.

The total space requirement is O(n).

Construction Time

- According to Corollary 2b, k can be found in expected time $O(n)$.
- W_i, b_i, m_i, s_i can be computed in time $O(n)$.
- According to Corollary 2a each k_i can be computed in expected time $O(b_i^2)$.

Total expected time:

$$O\left(n + \sum_{i=0}^{n} b_i^2\right) = O(n + B_k) = O(n)$$

Main Result

Theorem: Let N be a prime number and $S \subseteq U = [0 \ldots N\text{-}1]$ with $|S| = n$.
A perfect hash table of size O(n) and a hash function with access time $O(1)$ can be constructed in expected time O(n).

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