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Algorithms for Radio Networks

Fourier-Analysis and Modulation

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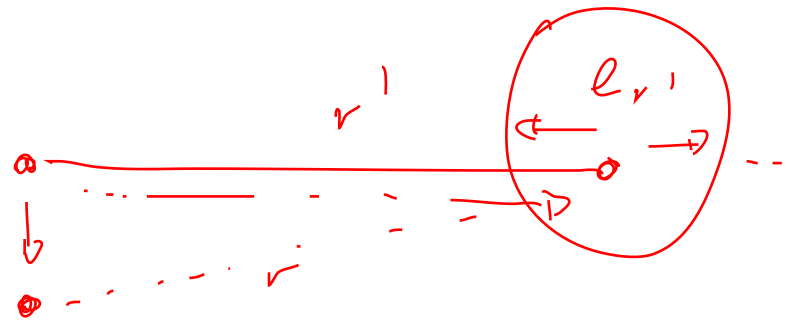
How does it really work?

- **Charged particles apply forces to other charge particles**
  - electrons (-) repel other electrons (-)
  - protons (+) attract electrons (-)
- **The possibility to apply a force to a particle is called field**
  - here electric field
- **The electric field of a particle depends on**
  - the charge q (+/-, strenght)
  - the distance from the particle r‘
    - respecting the speed of light
  - the direction towards the particle $e_{r'}$
  - the speed and the acceleration

q

$$\mathbf{E} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{e_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{\mathbf{r}'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{\mathbf{r}'}\right)$$



Radio Communication in the Far Distance

- **Superposition Principle:**
  - Electric Fields add up, since they represent forces
- **The dominant term in the distance is the acceleration term**
- **In radio communication electrons are moved by sinus curves**
  - so we can (sloppily) replace:

$$e_{r'} \approx \frac{1}{r'} a \sin 2\pi f t$$

  - with amplitude a and frequency f
- **The Energy P of an electric field**
  - is proportional to the square of the field E

$$\mathbf{E} = \frac{-q}{4\pi\varepsilon_0}\left(\cancel{\frac{e_{r'}}{r'^2}} + \cancel{\frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e_{r'}}}{r'^2}\right)} + \boxed{\frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e_{r'}}} \right)$$

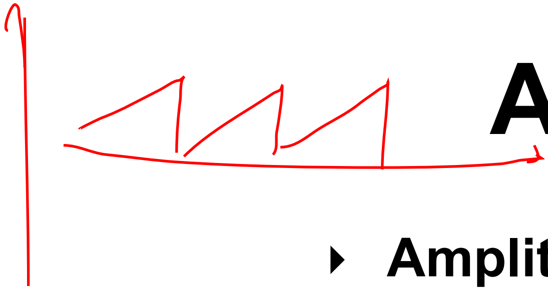
$\sin 2\pi f t$

$2\pi f \cdot \cos 2\pi f t$

$-(2\pi f)^2 \cdot \sin 2\pi f t$

r

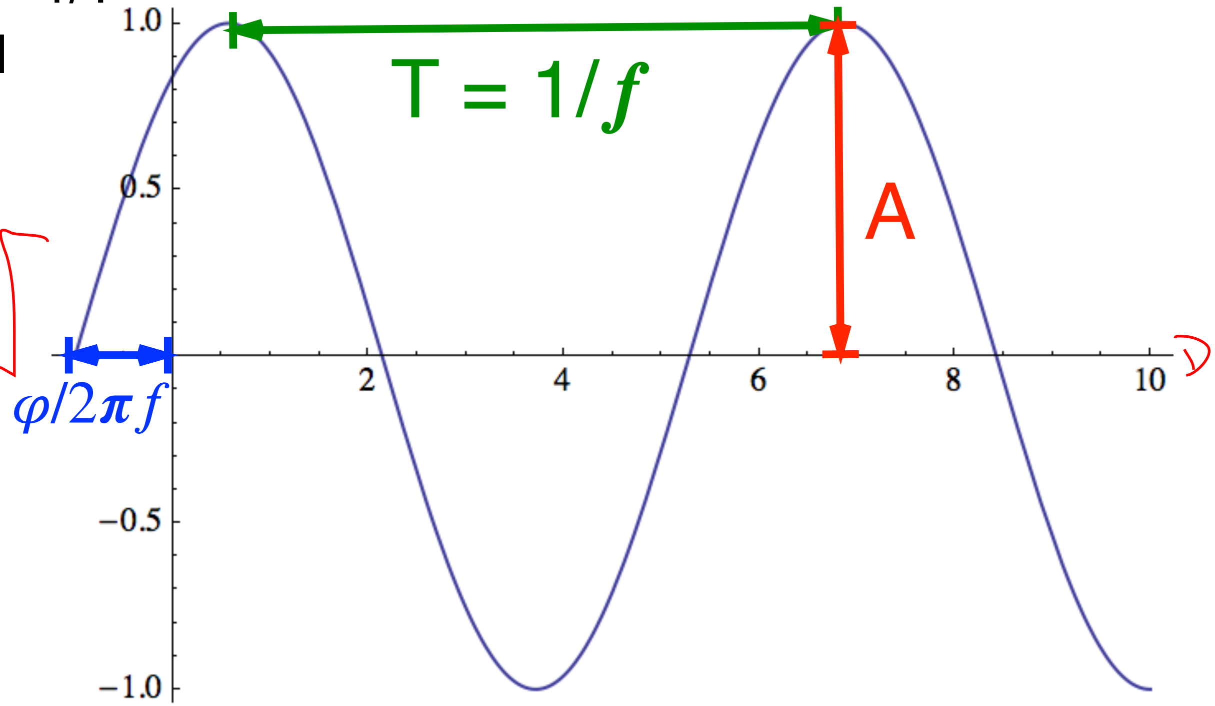
Amplitude Representation

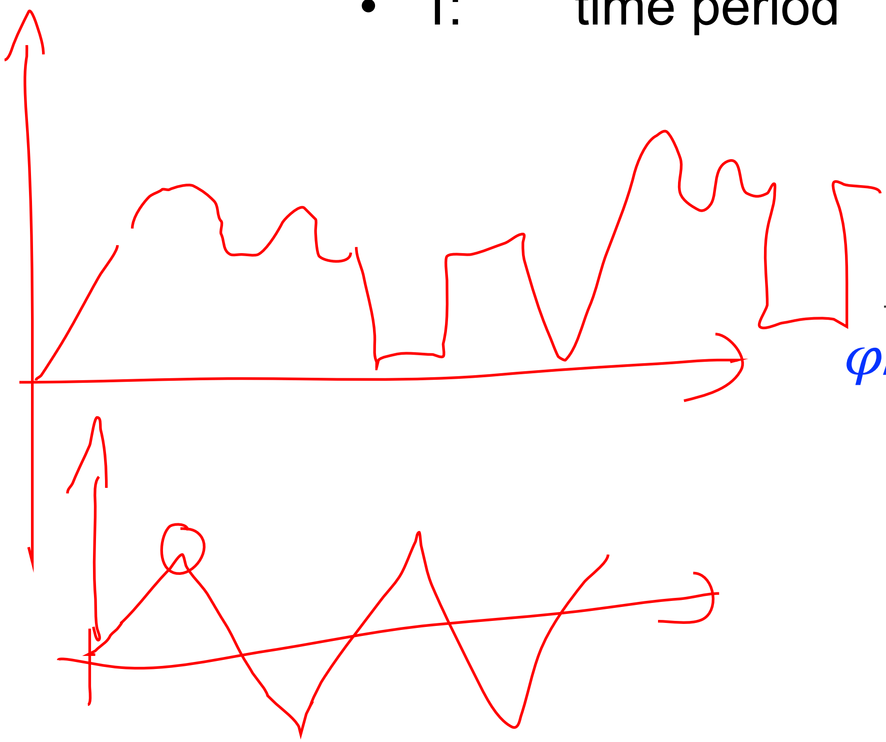


- **Amplitude representation of a sine curve**

$$s(t) = A \sin(2\pi f t + \phi)$$

  - A: amplitude
  - ϕ: phase shift
  - f : frequency = 1/T
  - T: time period

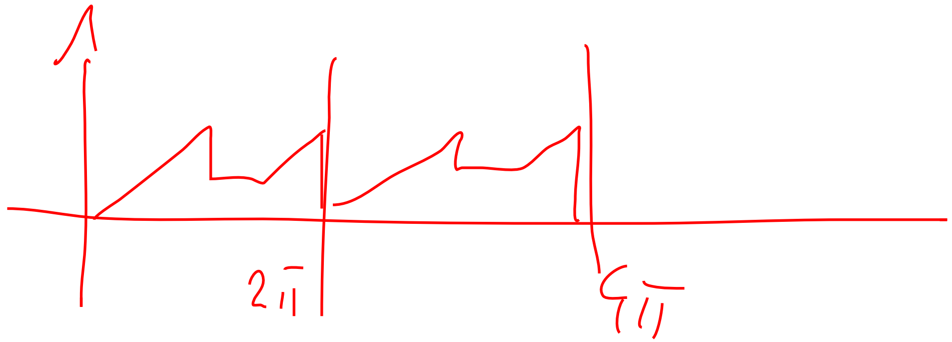




Fourier Transformation

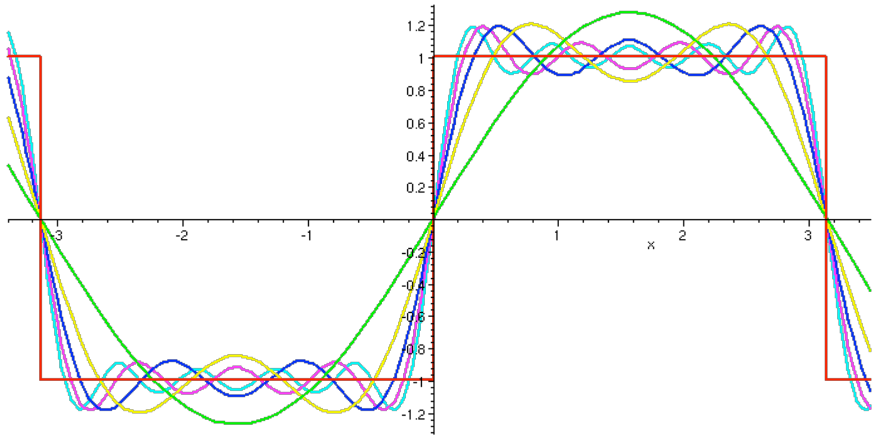
$\frac{a_0}{2} + \sum_k a_k \cos kx + b_k \sin kx$

- **Fourier transformation of a periodic function**
  - decomposition in various sine and cosine functions
- **Dirichlet condition of a periodic function f**
  - $f(x) = f(x+2\pi)$
  - $f(x)$ in $(-\pi,\pi)$ in finitely many intervals continuous and monotonic
  - If f is discontinuous at x_0, then $f(x_0)=(f(x_0-0)+f(x_0+0))/2$



- **Theorem of Dirichlet:**
  - If $f(x)$ satisfies $(-\pi,\pi)$ the Dirichlet condition then there exists Fourier coefficients $a_0,a_1,a_2,\ldots,b_1,b_2,\ldots$ such that

$$\lim_{n\to\infty} \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx = f(x) .$$



$\frac{\sin x}{2}$

Computation of Fourier Coefficients

$\pi - 2 \sin x$

$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx$

- **Fourier coefficients a_i, b_i:**
  - For k = 0,1,2,…

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \; dx$$

  - For k = 1,2,3,…

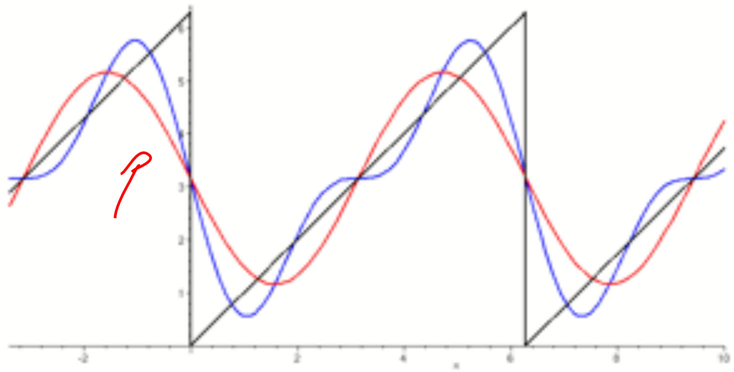
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \; dx$$

- **Example: saw tooth curve**

$$f(x) = x \text{ , für } 0 < x < 2\pi$$

$$f(x) = \pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

$\pi - 2 \frac{\sin x}{1} + \frac{2}{2} \sin 2x + \frac{2}{3} \sin 3x$



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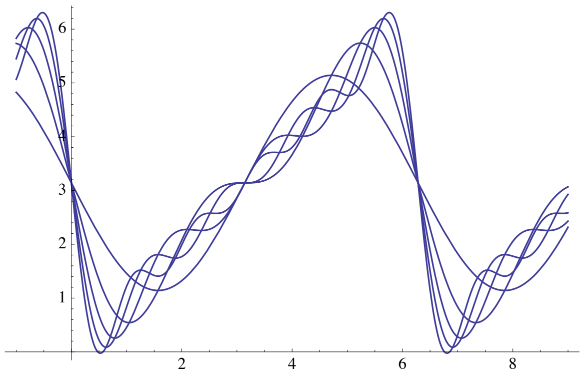
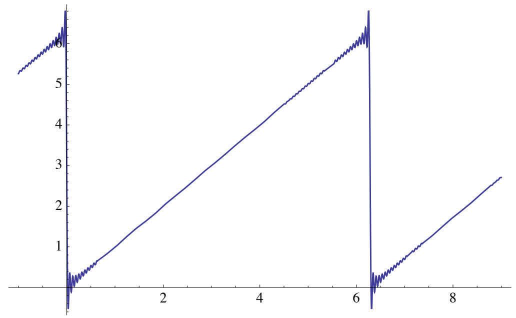




Fourier Analysis for General Period

- **Theorem of Fourier for period T=1/f:**
  - The coefficients c, a_n, b_n are then obtained as follows

$$g(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi k f t) + b_k \sin(2\pi k f t)$$

$$a_k = \frac{2}{T} \int_0^T g(t) \cos(2\pi n f t) dt$$

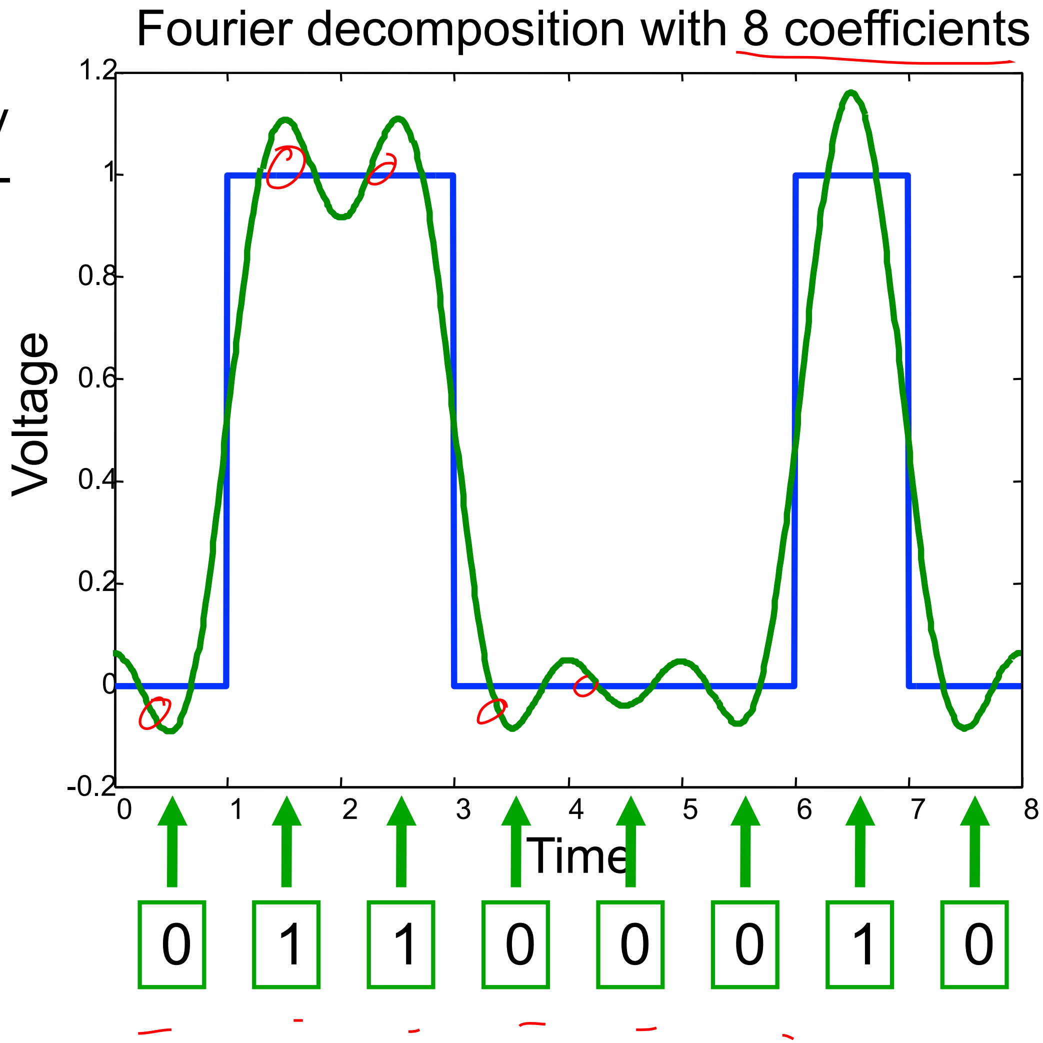
$$b_k = \frac{2}{T} \int_0^T g(t) \sin(2\pi n f t) dt$$

- **The sum of squares of the k-th terms is proportional to the energy consumed in this frequency:**

$$(a_k)^2 + (b_k)^2$$

How often do you measure?

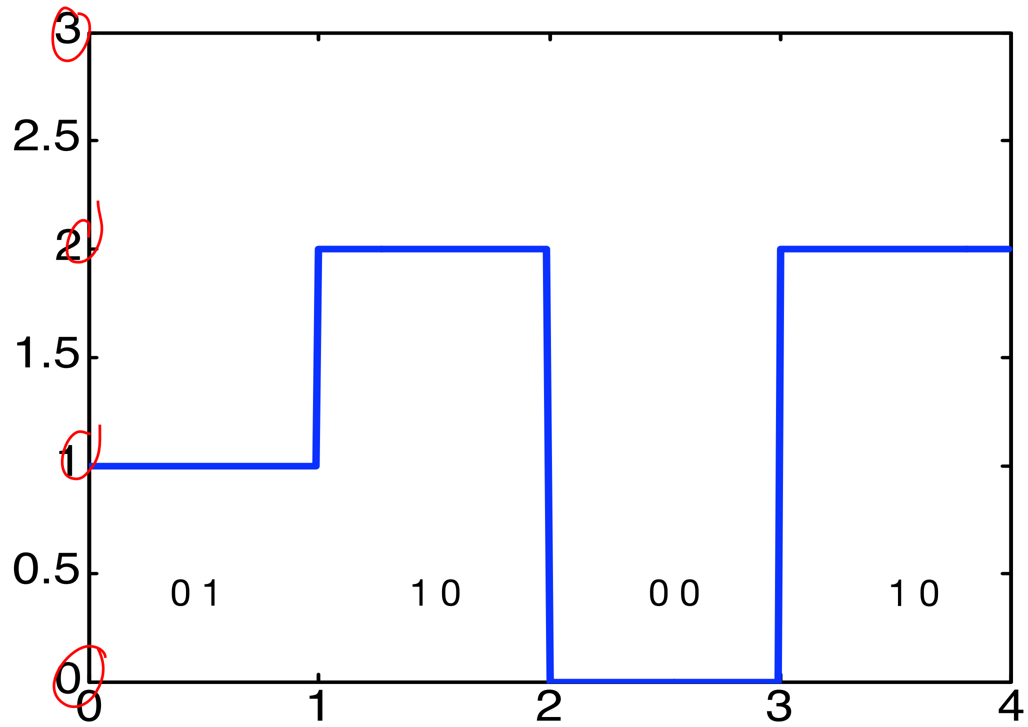
- How many measurements are necessary to determine a Fourier transform to the k-th component, exactly?
- **Nyquist-Shannon sampling theorem**
  - To reconstruct a continuous band-limited signal with a maximum frequency f_{max} you need at least a sampling frequency f max of 2 f_{max}.



Symbols and Bits

- **For data transmission instead of bits can also be used symbols**
  - E.g. 4 Symbols: A, B, C, D with
    - A = 00, B = 01, C = 10, D = 11
- **Symbols**
  - Measured in baud
  - Number of symbols per second
- **Data rate**
  - Measured in bits per second (bit / s)
  - Number of bits per second
- **Example**
  - 2400 bit/s modem is 600 baud (uses 16 symbols)

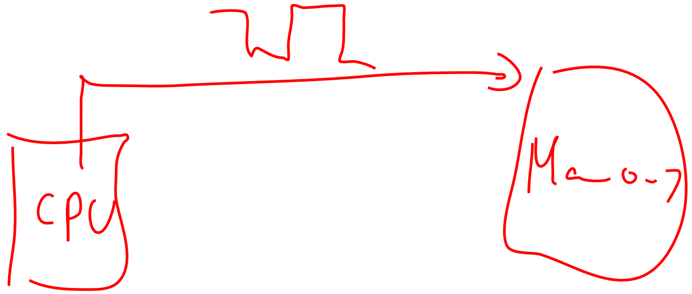
$2^2 = 4$

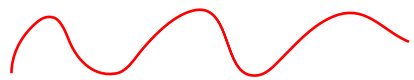


$16 = 2^4$

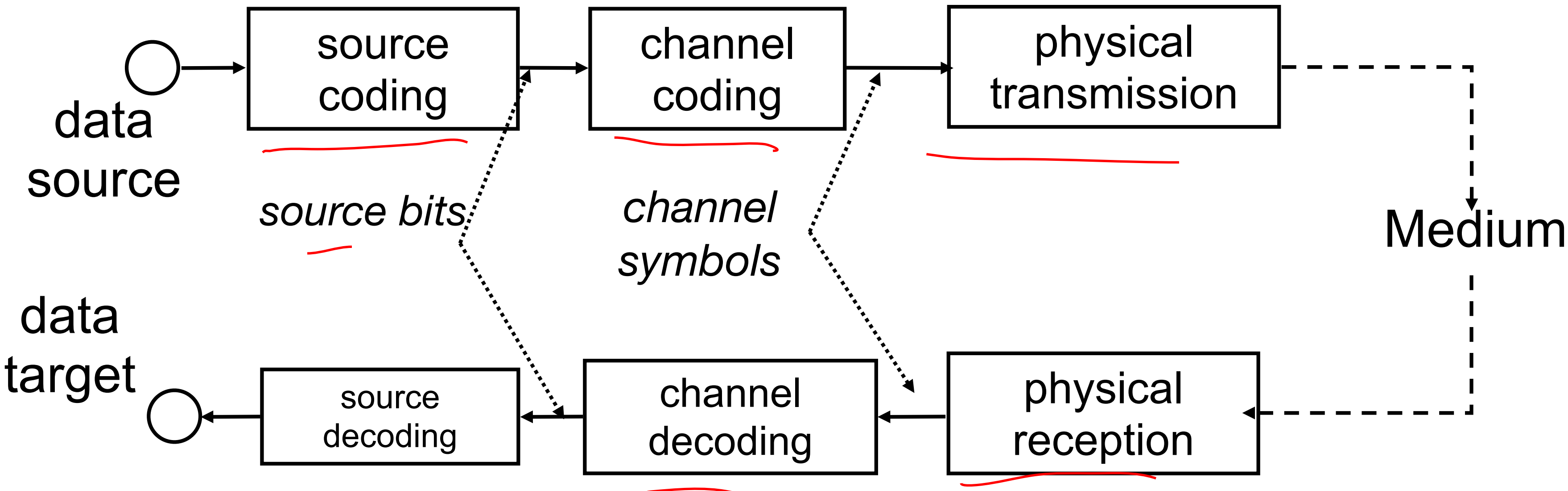
$256 = 2^8$

Structure of a *Baseband* Digital Transmission



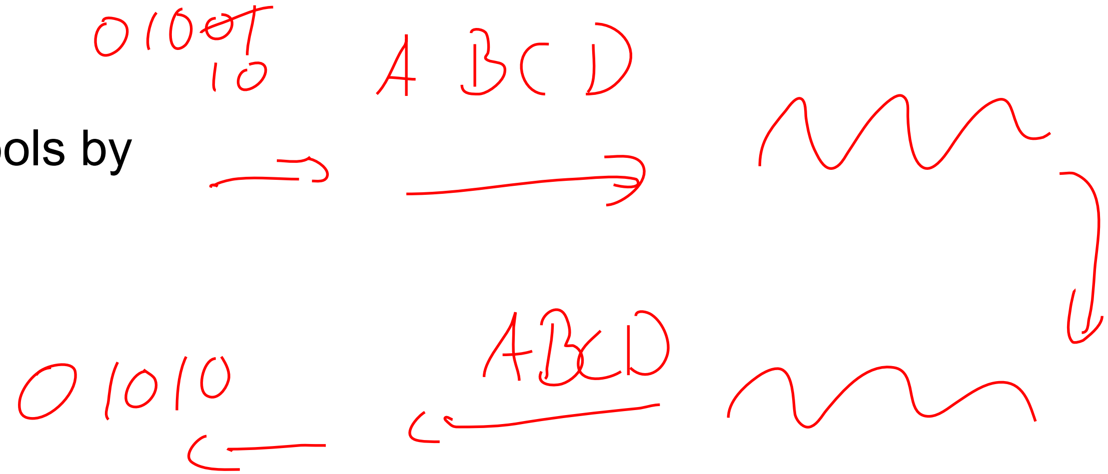


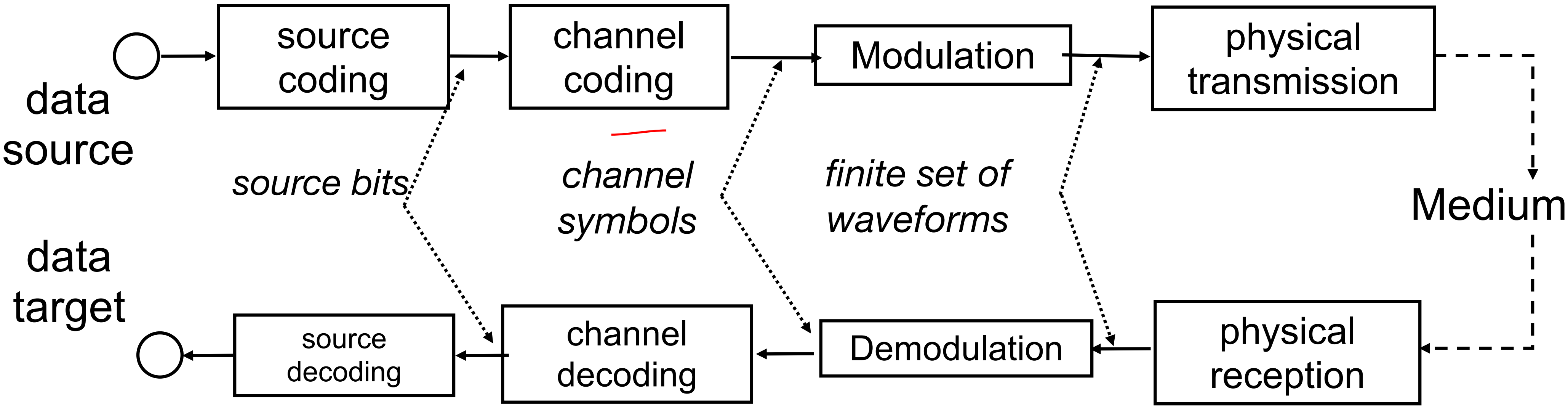
- **Source Coding**
  - removing redundant or irrelevant information
  - e.g. with lossy compression (MP3, MPEG 4)
  - or with lossless compression (Huffman code)
- **Channel Coding**
  - Mapping of source bits to channel symbols
  - Possibly adding redundancy adapted to the channel characteristics
  - physical transmission
- **Conversion into physical events**



Structure of a *Broadband* Digital transmission

- **MOdulation/DEModulation**
  - Translation of the channel symbols by
    - amplitude modulation
    - phase modulation
    - frequency modulation
    - or a combination thereof

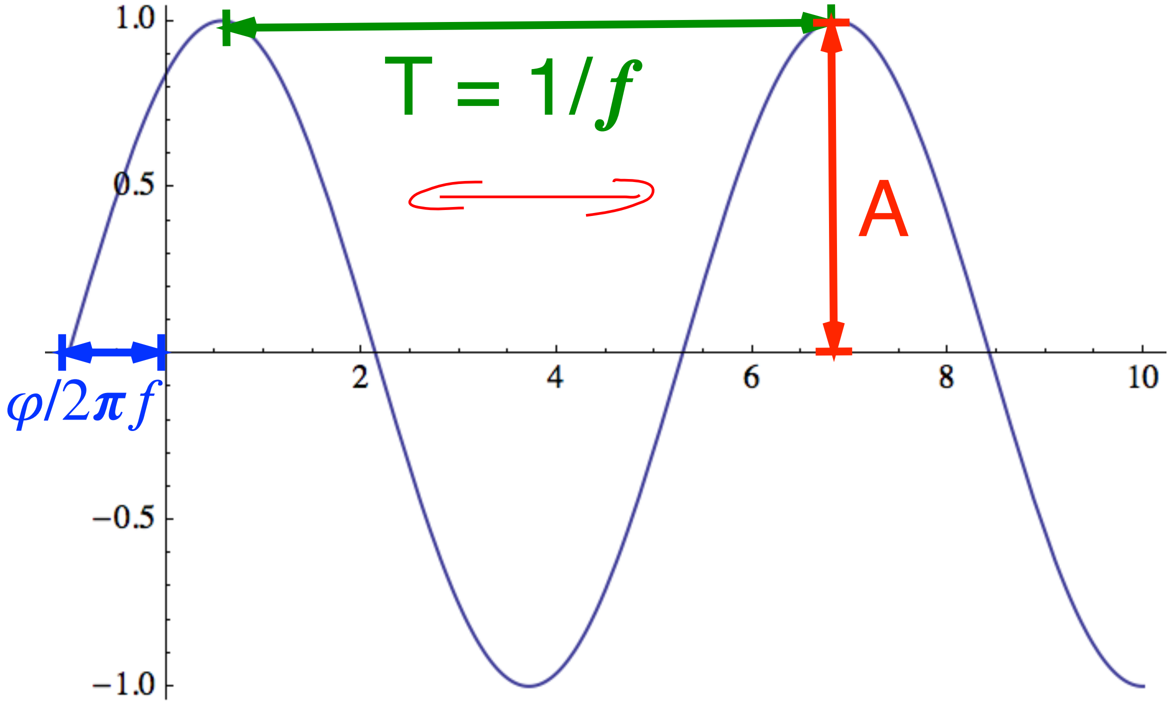




Broadband

- **Idea**
  - Focusing on the ideal frequency of the medium
  - Using a sine wave as the carrier wave signals
- **A sine wave has no information**
  - the sine curve continuously (modulated) changes for data transmission,
  - implies spectral widening (more frequencies in the Fourier analysis)
- **The following parameters can be changed:**
  - Amplitude A
  - Frequency f=1/T
  - Phase φ

$$s(t) = A\sin(2\pi f t + \phi)$$

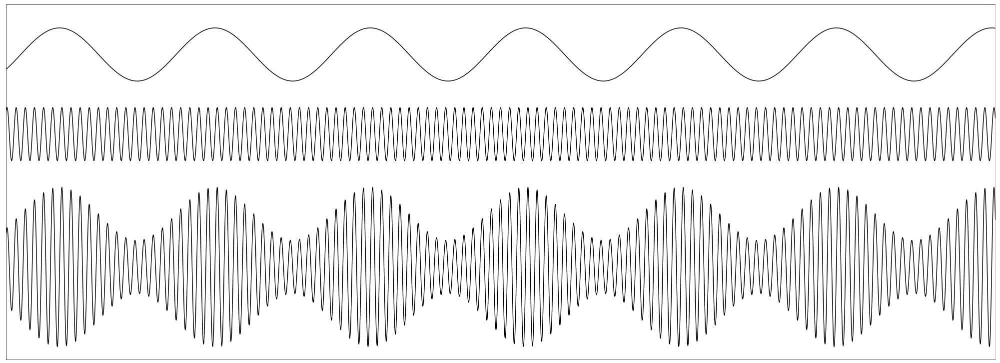


Amplitude Modulation

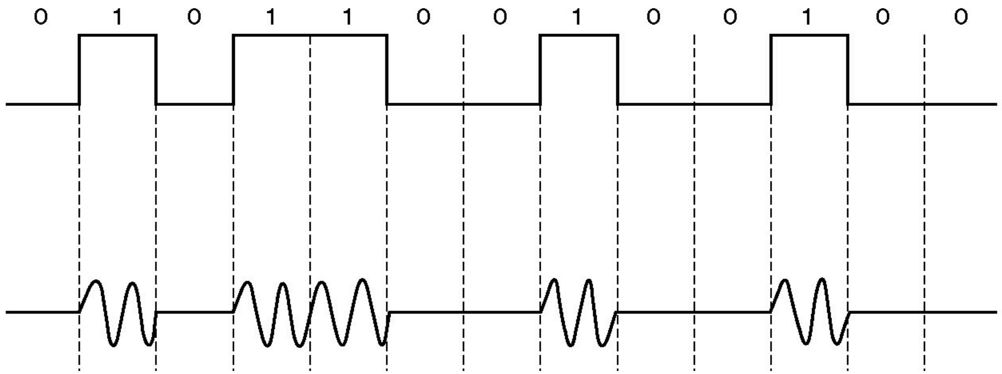
- **The time-varying signal s (t) is encoded as the amplitude of a sine curve:**

$$f_A(t) = s(t)\,\sin(2\pi f t + \phi)$$

- **Analoges Signal**
  - analog signal
  - amplitude modulation
  - Continuous function in time
  - e.g. second prolonged wave signal (sound waves)



- **Digital signal**
  - amplitude keying
  - E.g. given by symbols as a symbol of strength
  - special case: symbols 0 or 1
    - on / off keying

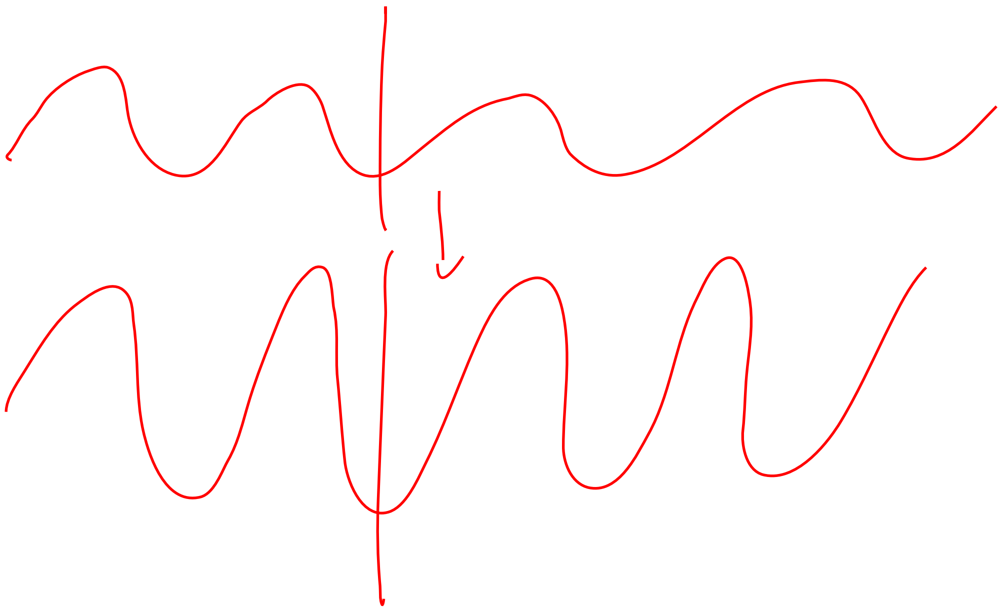


Amplitude Shift Keying (ASK)

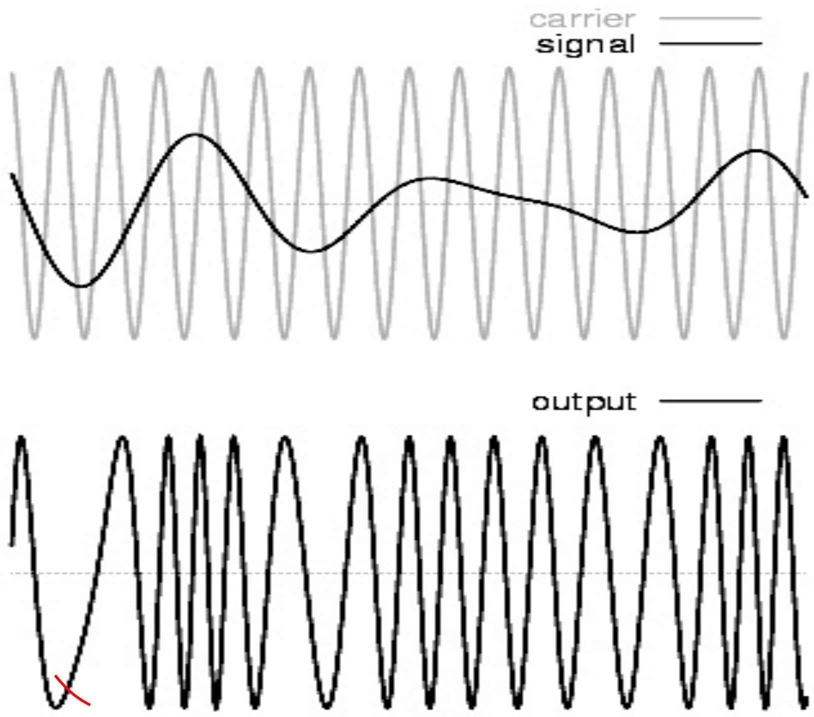
- **Let $E_i(t)$ is the symbol energy at time t**

$$s_i(t) = \sqrt{\frac{2E_i(t)}{T}} \cdot \sin(\omega_0 t + \phi)$$

- **Example: $E_0(t) = 1$, $E_1(t) = 2$**



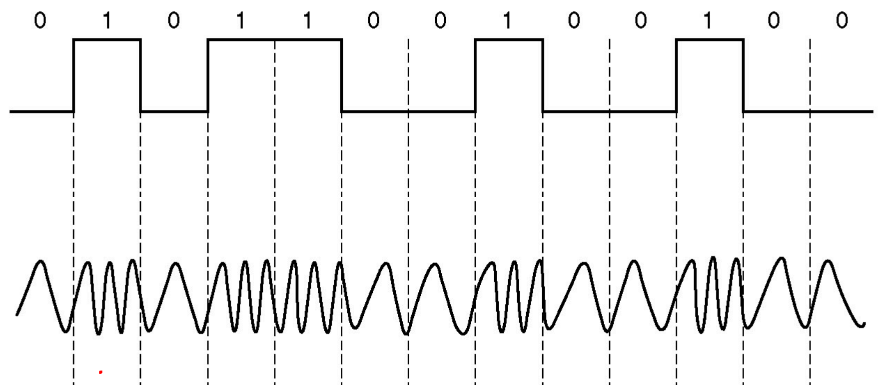
Frequency Modulation



- **The time-varying signal s (t) is encoded in the frequency of the sine curve:**

$$f_F(t) = a \sin(2\pi s(t) t + \phi)$$

- **Analog signal**
  - Frequency modulation (FM)
  - Continuous function in time
- **Digital signal**
  - Frequency Shift Keying (FSK)
  - E.g. frequencies as given by symbols

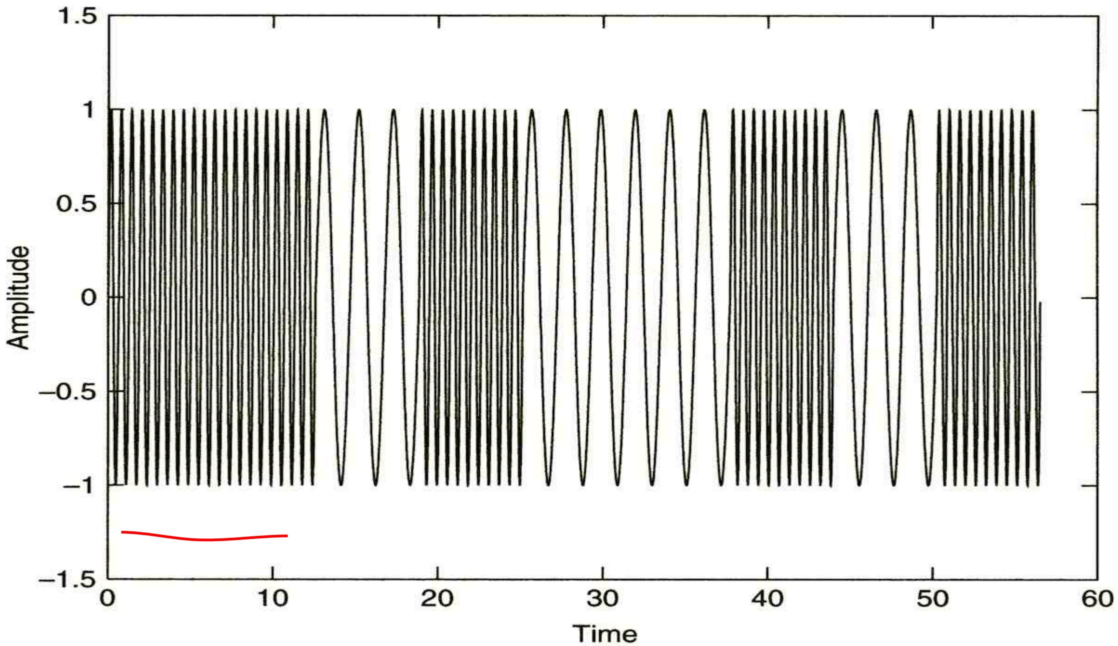


Frequency Shift Keying (FSK)

- **Frequency signals $\omega_i(t)$**

$$s_i(t) = \sqrt{\frac{2E}{T}} \cdot \sin(\omega_i(t) \cdot t + \phi)$$

- **Example:**

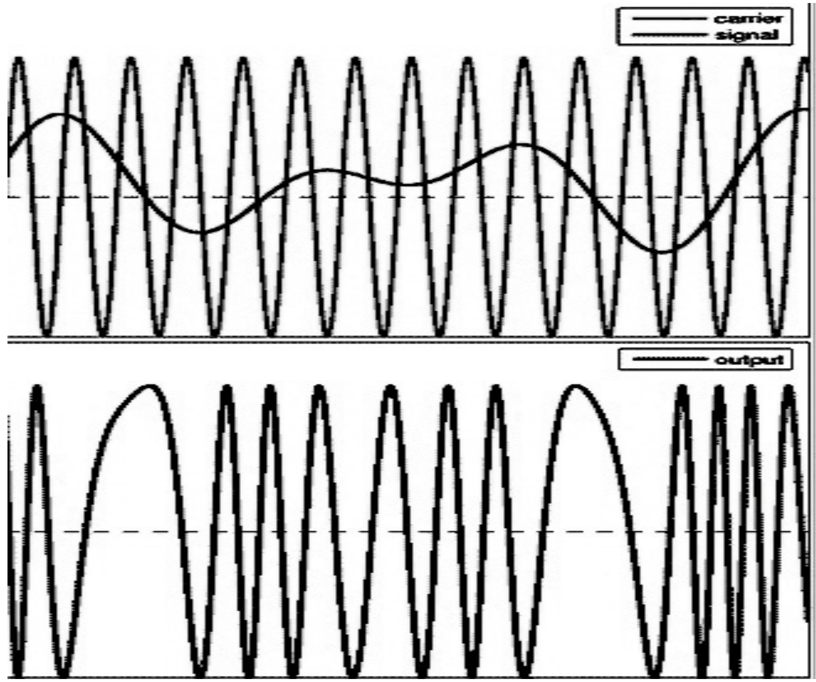


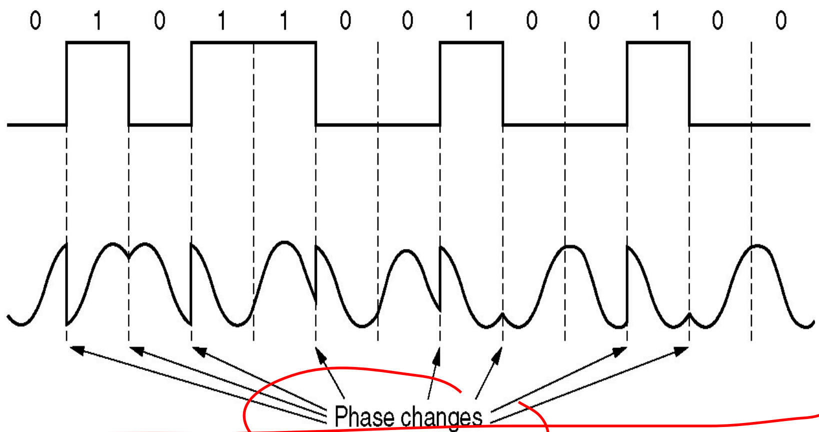
Phase Modulation

- **The time-varying signal s (t) is encoded in the phase of the sine curve:**

$$f_P(t) = a\,\mathsf{sin}(2\pi f t + s(t))$$

- **Analog signal**
  - phase modulation (PM)
  - very unfavorable properties
  - es not used
- **Digital signal**
  - phase-shift keying (PSK)
  - e.g. given by symbols as phases





Digital and Analog signals in Comparison

- **For a station there are two options**
  - digital transmission
    - finite set of discrete signals
    - e.g. finite amount of voltage sizes / voltages
  - analog transmission
    - Infinite (continuous) set of signals
    - E.g. Current or voltage signal corresponding to the wire
- **Advantage of digital signals:**
  - There is the possibility of receiving inaccuracies to repair and reconstruct the original signal
  - Any errors that occur in the analog transmission may increase further

Phase Shift Keying (PSK)

00	01	10	11
$\frac{\pi}{4}$	$\frac{3}{4}\pi$	$\frac{5}{4}\pi$	$\frac{7}{4}\pi$

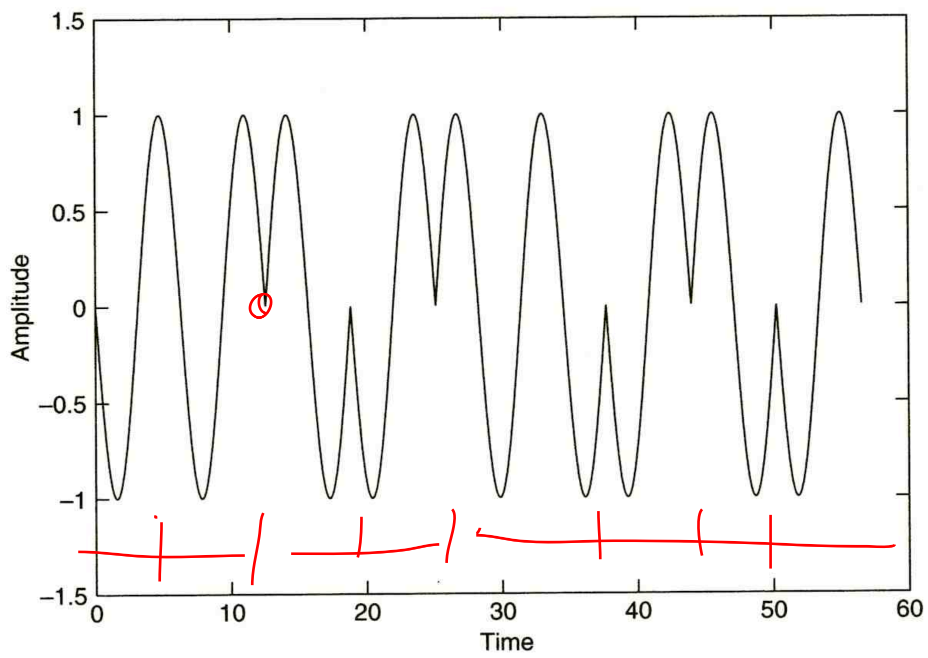
- For phase signals $\phi_i(t)$

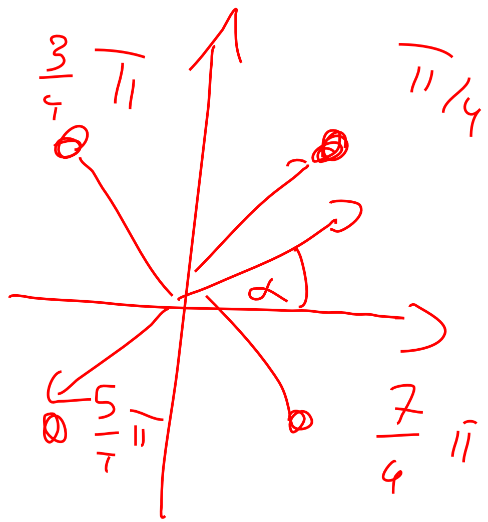
$$s_i(t) = \sqrt{\frac{2E}{T}} \cdot \sin(\omega_0 t + \phi_i(t))$$

2π

$0, 2\pi$

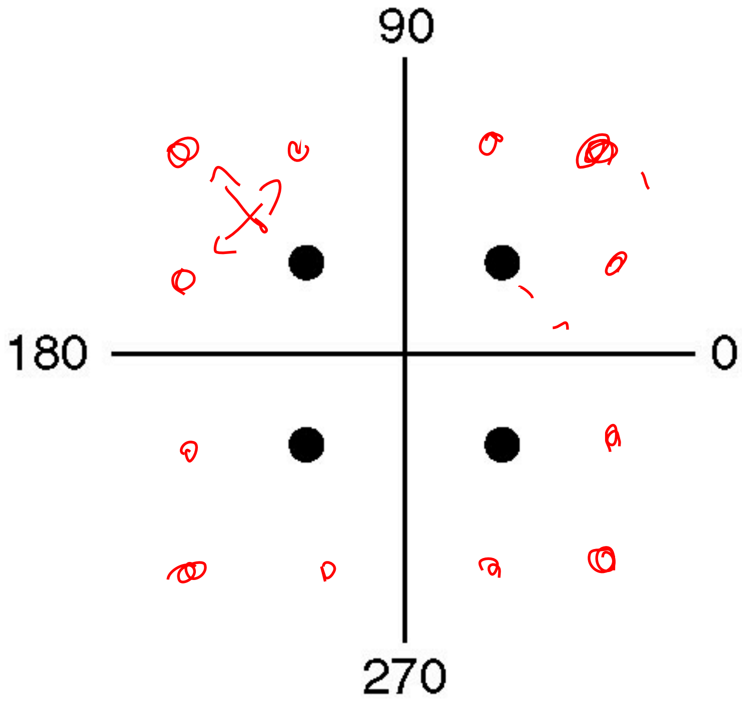
- Example:





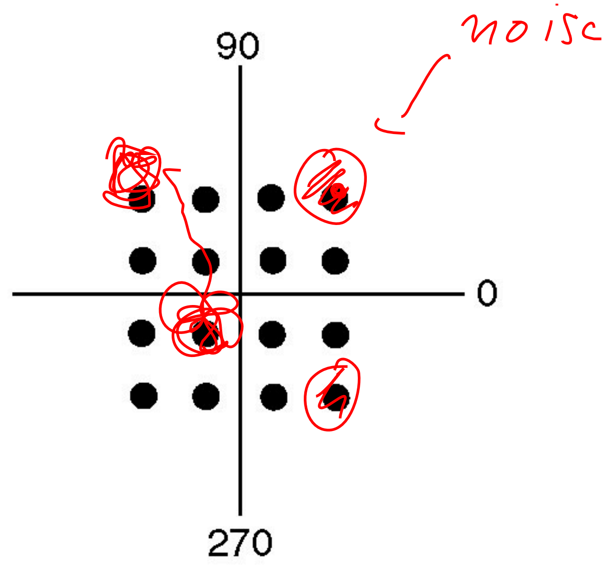
PSK with Different Symbols

- **Phase shifts can be detected by the receiver very well**
- **Encoding various Symoble very simple**
  - Using phase shift e.g. π / 4, 3/4π, 5/4π, 7/4π
    - rarely: phase shift 0 (because of synchronization)
  - For four symbols, the data rate is twice as large as the symbol rate
- **This method is called Quadrature Phase Shift Keying (QPSK)**



Amplitude and Phase Modulation

- **Amplitude and phase modulation can be successfully combined**
  - Example: 16-QAM (Quadrature Amplitude Modulation)
    - uses 16 different combinations of phases and amplitudes for each symbol
    - Each symbol encodes four bits ($2^4 = 16$)
  - The data rate is four times as large as the symbol rate



Nyquist's Theorem

- **Definition**
  - The band width H is the maximum frequency in the Fourier decomposition
- **Assume**
  - The maximum frequency of the received signal is f = H in the Fourier transform
    - (Complete absorption [infinite attenuation] all higher frequencies)
  - The number of different symbols used is V
  - No other interference, distortion or attenuation of
- **Nyquist theorem**
  - The maximum symbol rate is at most 2 H baud.
  - The maximum possible data rate is a bit more than ~~$2 \log_2 H V$~~ / s.

$2 H \log_2 V$

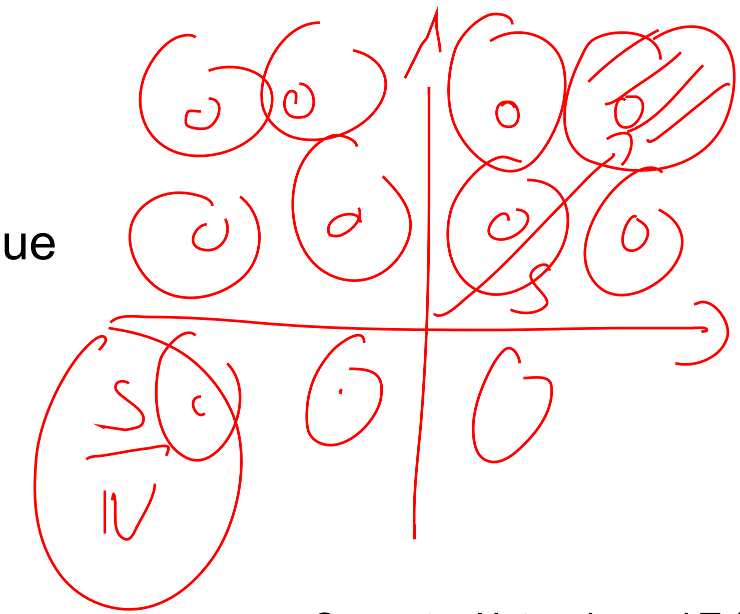
Do more symbols help?

- **Nyquist's theorem states that could theoretically be increased data rate with the number of symbols used**
- **Discussion:**
  - Nyquist's theorem provides a theoretical upper bound and no method of transmission
  - In practice there are limitations in the accuracy
  - Nyquist's theorem does not consider the problem of noise

The Theorem of Shannon

- **Indeed, the influence of the noise is fundamental**
  - Consider the relationship between transmission intensity S to the strength of the noise N
  - The less noise the more signals can be better recognized
- **Theorem of Shannon**
  - The maximum possible data rate is $\boxed{H \log_2 (1 + S / N) \text{ bits / s}}$
    - with bandwidth H
    - Signal strength S
- **Attention**
  - This is a theoretical upper bound
  - Existing codes do not reach this value

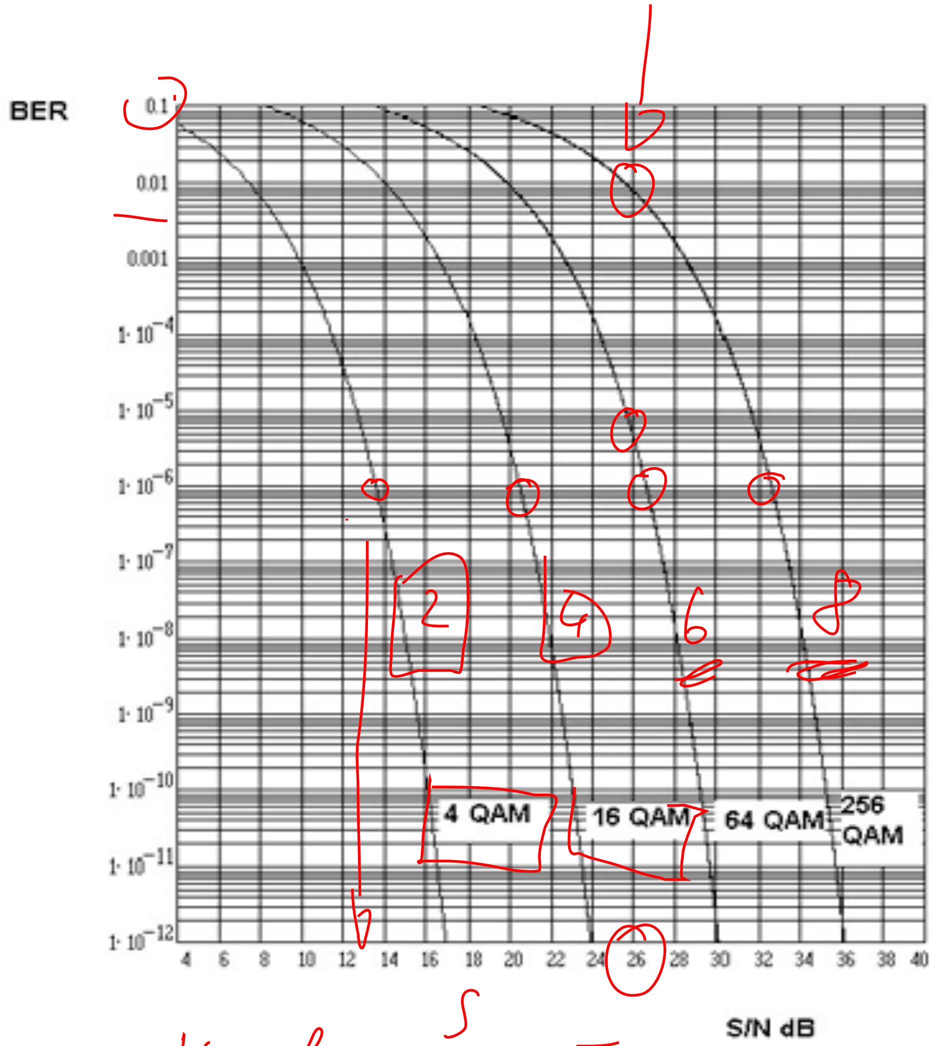
N



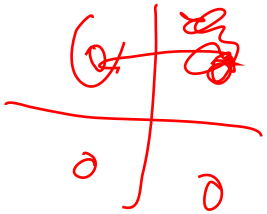
$\log\left(\frac{S}{N} + 1\right)$

Bit Error Rate and SINR

- **Higher SIR decreases Bit Error Rate (BER)**
  - BER is the rate of faulty received bits
- **Depends from the**
  - signal strength
  - noise
  - bandwidth
  - encoding
- **Relationship of BER and SINR**
  - Example: 4 QAM, 16 QAM, 64 QAM, 256 QAM



$10 \cdot \log_{10} \frac{S}{N} =$



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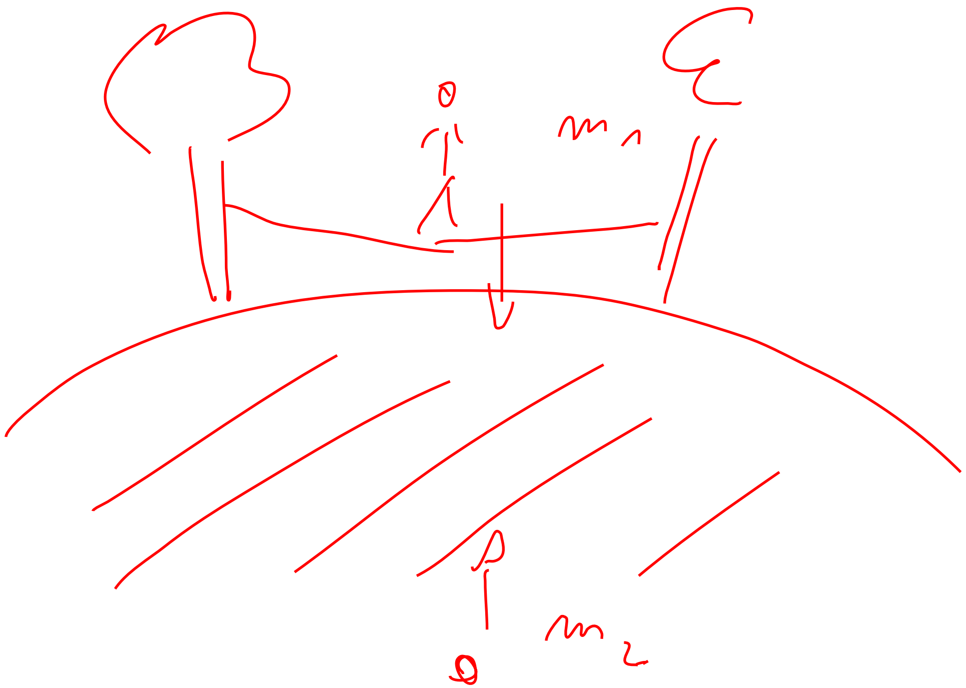
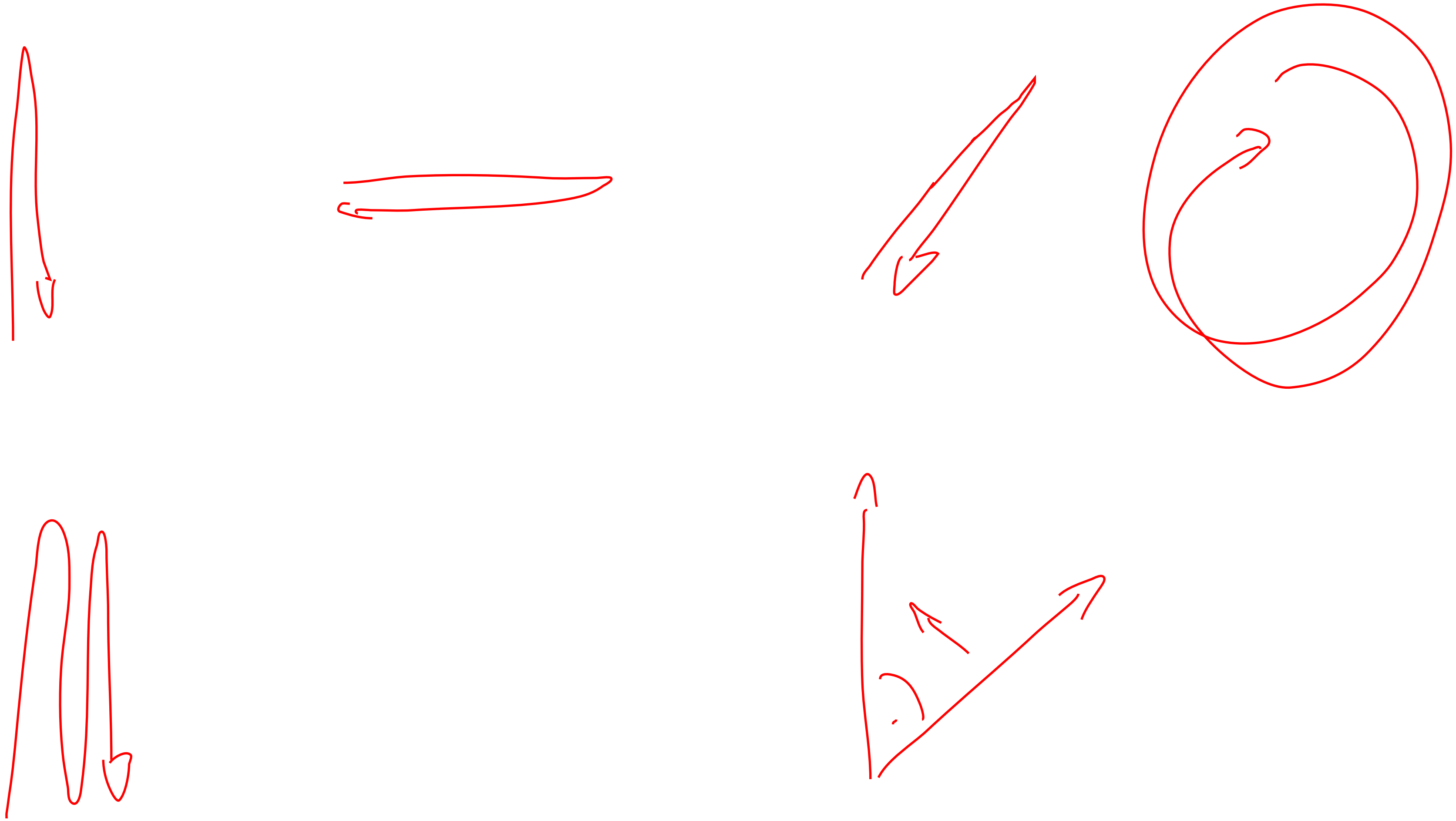
$\longleftarrow \bar{O}$ charge $\qquad\qquad \bar{O} \longrightarrow$



$m_1 \cdot m_2$



v

Speed ~ force

$\frac{1}{2} m v^2$

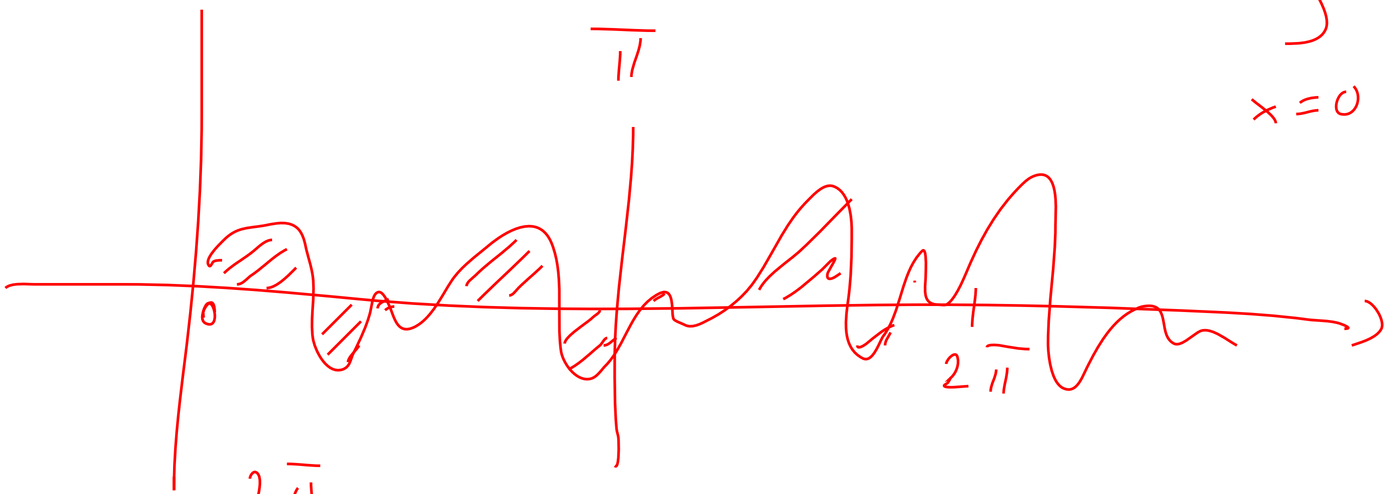
Energy ~ force^2

~ Field^2

$$\sum \sin kx + \cos kx$$

$$\int_{x=0}^{2\pi} (\sin kx) \cdot (\cos k' x)\, dx = 0 \qquad k \neq k'$$

$$\int_{x=0}^{2\pi} \sin kx \cos kx \, dx = 0$$



$$\int_{x=0}^{2\pi} \underline{\sin kx \cdot \sin kx}\, dx = \pi$$