ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG

Algorithms for Radio Networks

Network Coding

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IIF

Data Flows in Networks

- **Motivation**
  - Optimize data flow from source to target
- **Definition:**
  - (Single-commodity) maximum flow problem
  - Given
    - a graph G=(V,E)
    - a capacity function $w: E \to R^+_0$,
    - source set S and target set T
  - Find a maximum flow from S to T
- **A flow is a function $f: E \to R_0^+$ such that**
  - for all $e \in E$: $f(e) \leq w(e)$
  - for all $e \notin E$: $f(e) = 0$
  - for all $u,v \in V$: $f(u,v) \geq 0$

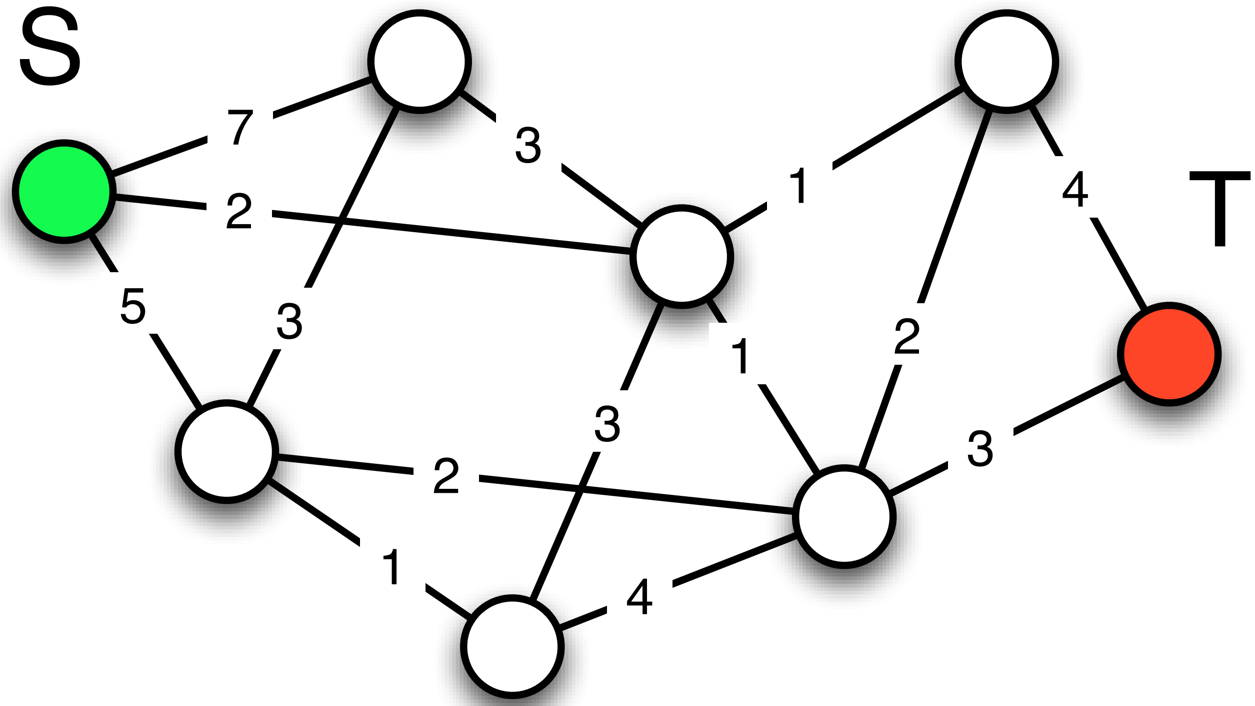
$$\forall u \in V \setminus (S \cup T)$$

$$\sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$$

- **Maximize flow**

$$\sum_{u \in S} \sum_{v \in V} f(u,v)$$

Data Flows in Networks



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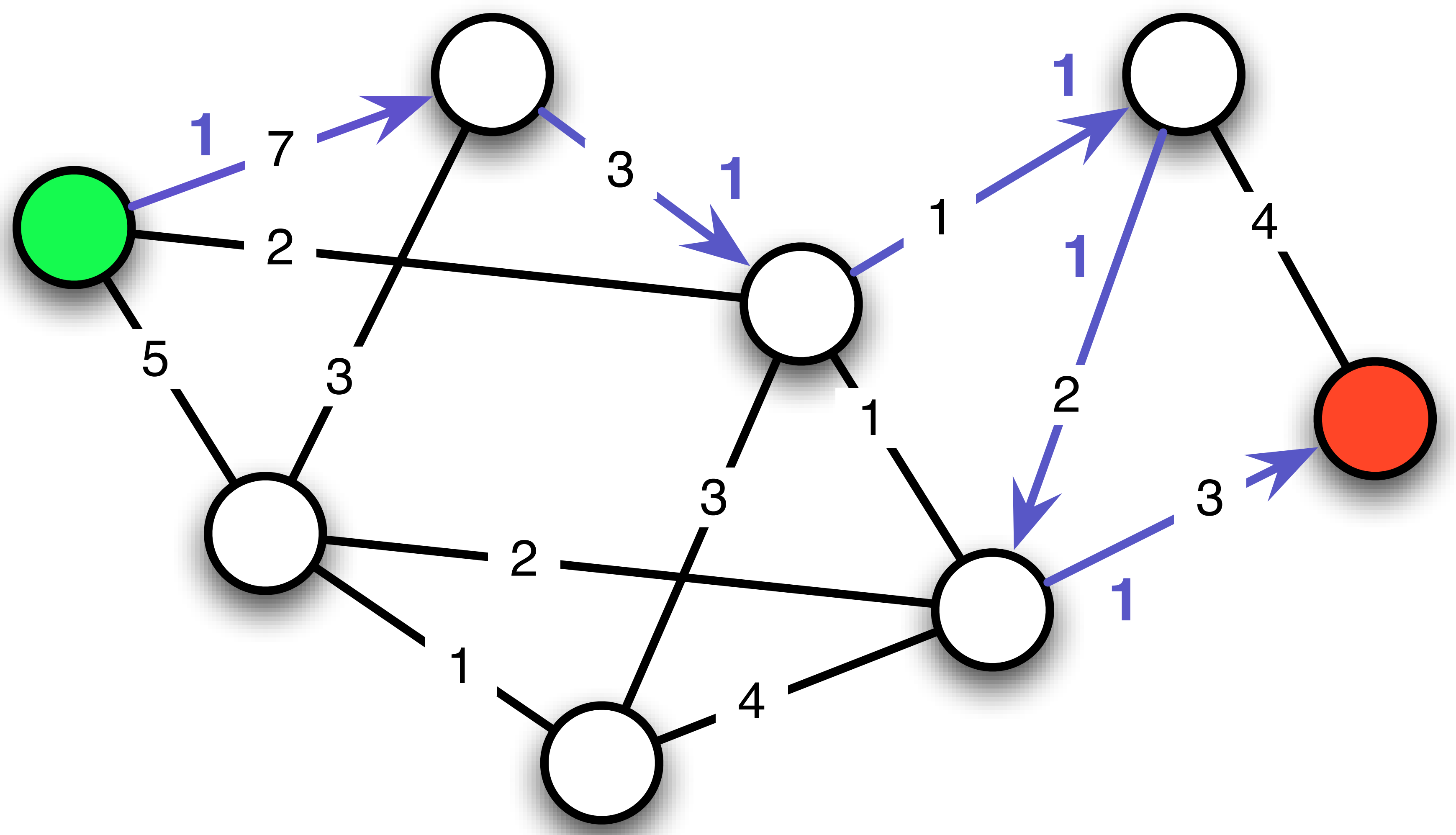

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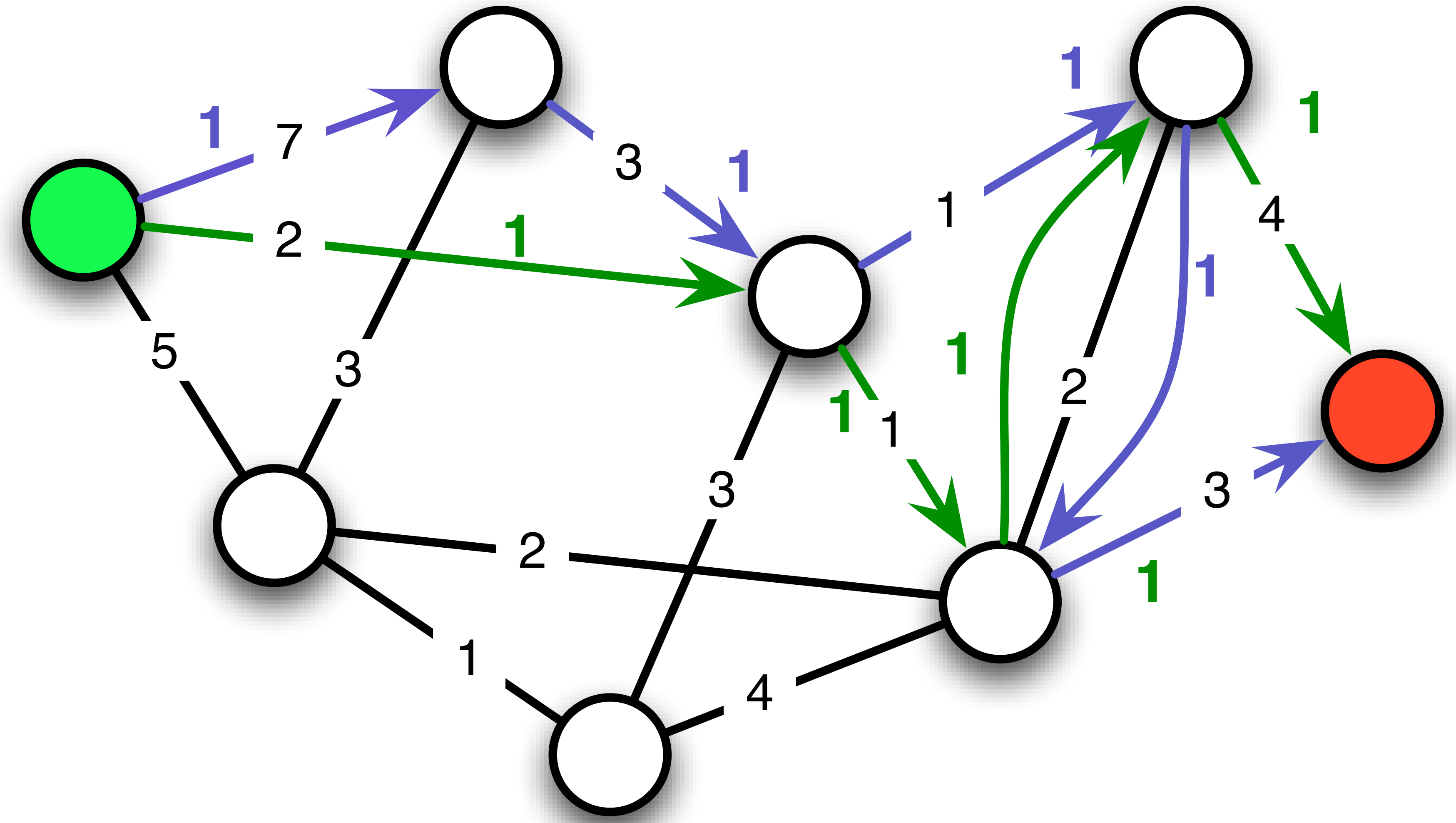

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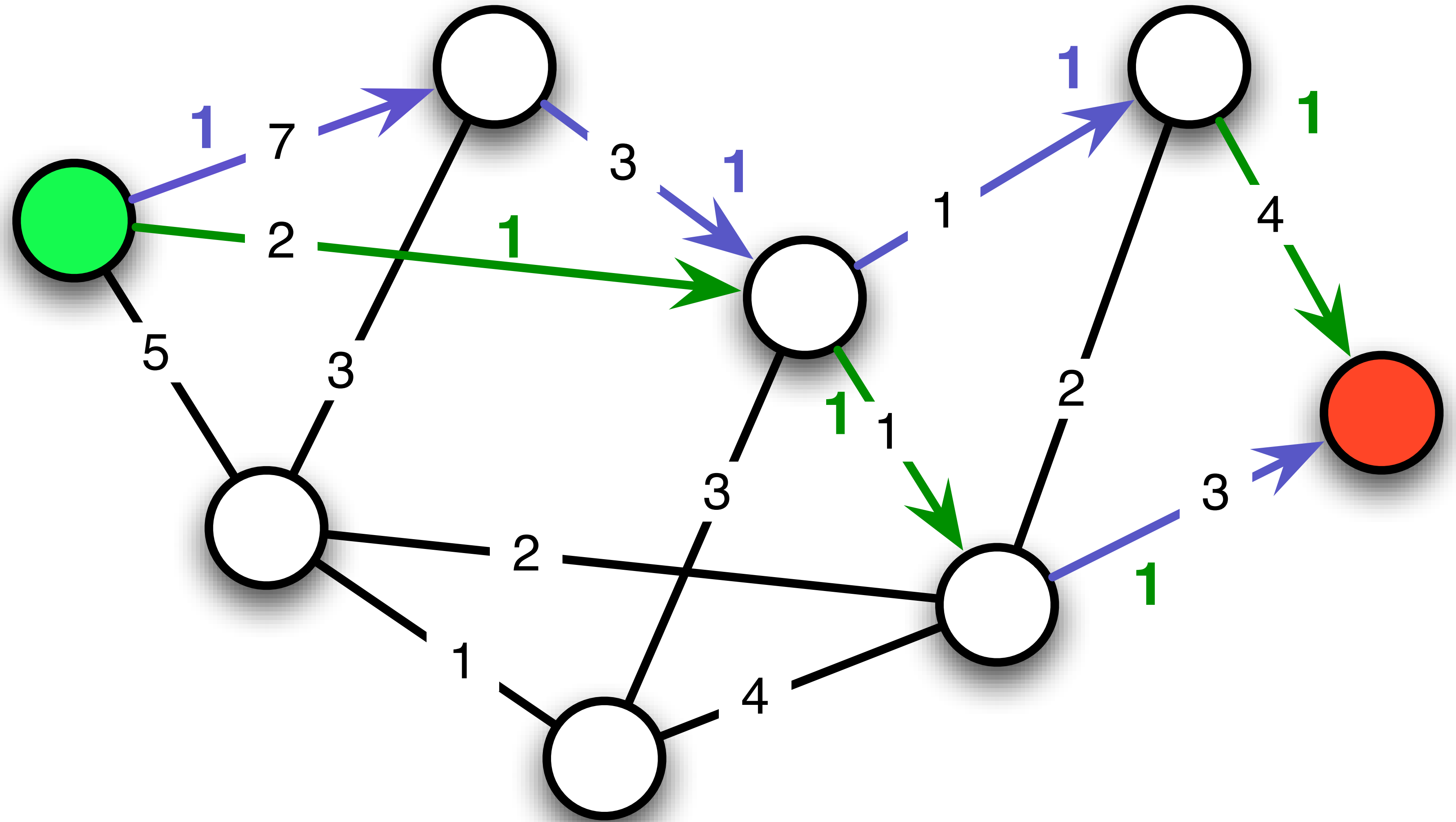
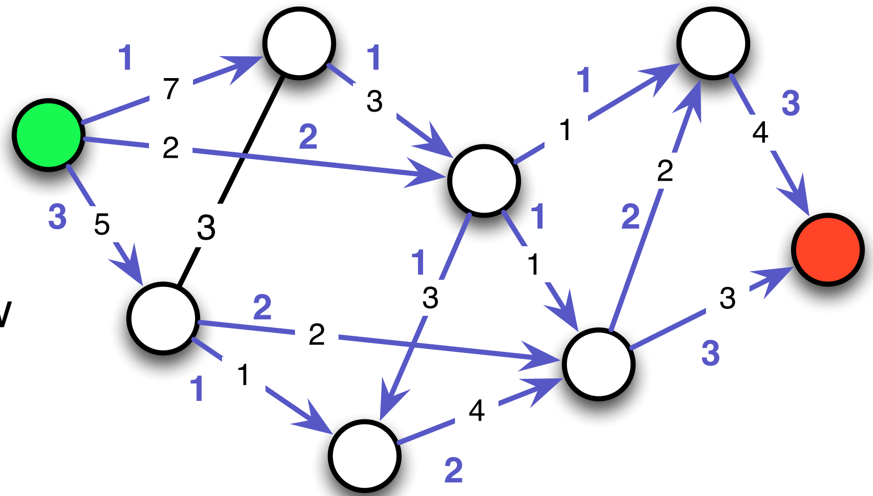


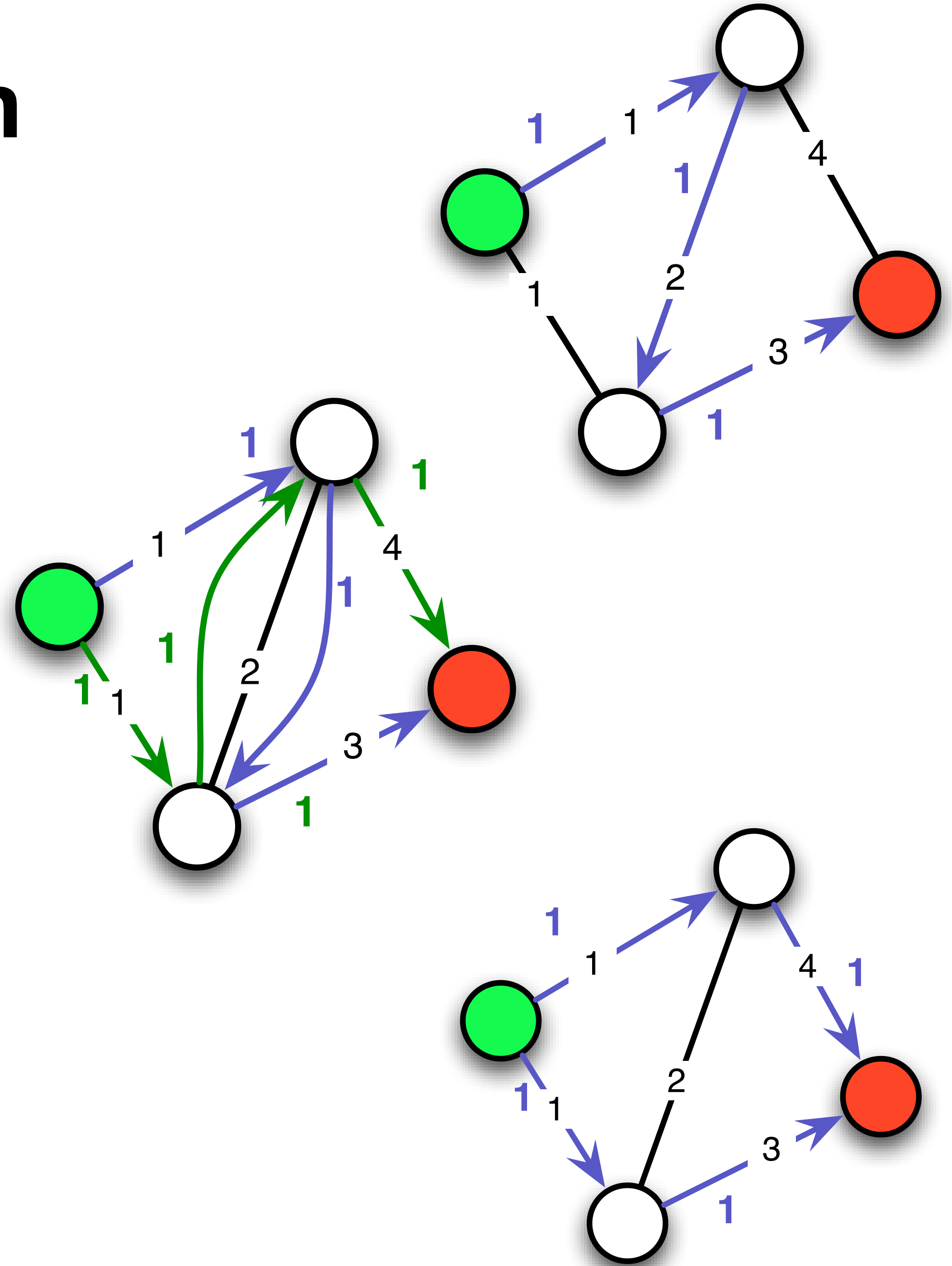
Computation of the Maximum Flow

- **Every natural pipe system solves the minimummaximum flow problem**
- **Algorithms**
  - Linear Programming
    - for real numbers
    - the flow is described by equations of a linear optimization problem
    - Simplex algorithm (or Ellipsoid method) can solve any linear equation system
  - Ford-Fulkerson
    - also for integers
    - as long as open paths exist, increase the flow on theses paths
      - open path: path which increases the flow
  - Edmonds-Karp
    - special case of Ford-Fulkerson
    - use BFS (breadth first search) to find open paths



Ford-Fulkerson

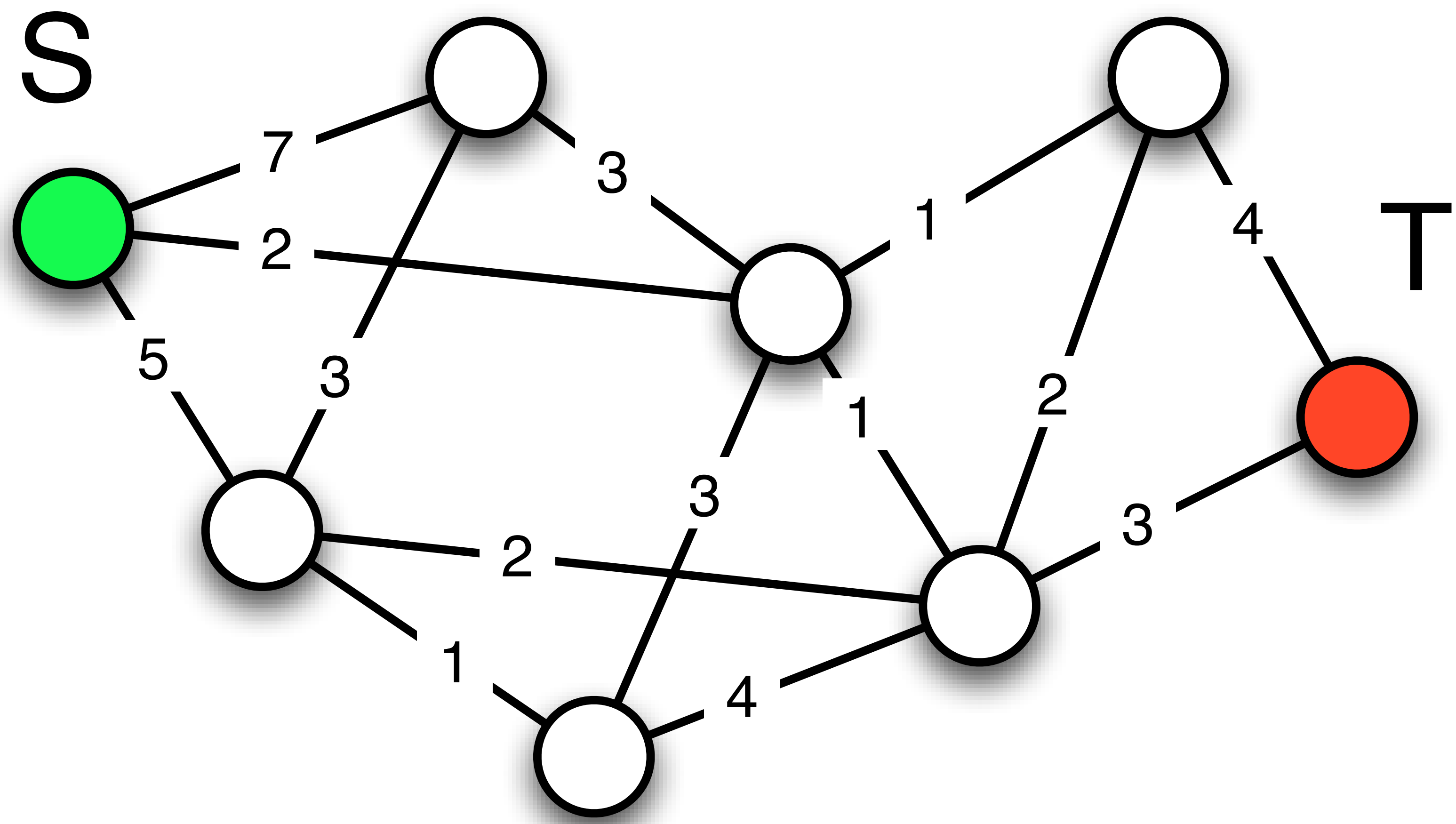
- **Find a path from the source node to the target node**
  - where the capacity is not fully utilized
  - or which reduces the existing flow
- **Compute the maximum flow on this augmenting path**
  - by the minimum of the flow that can be added on all paths
- **Add the flow on the path to the existing flow**
- **Repeat this step until no flow can be added anymore**



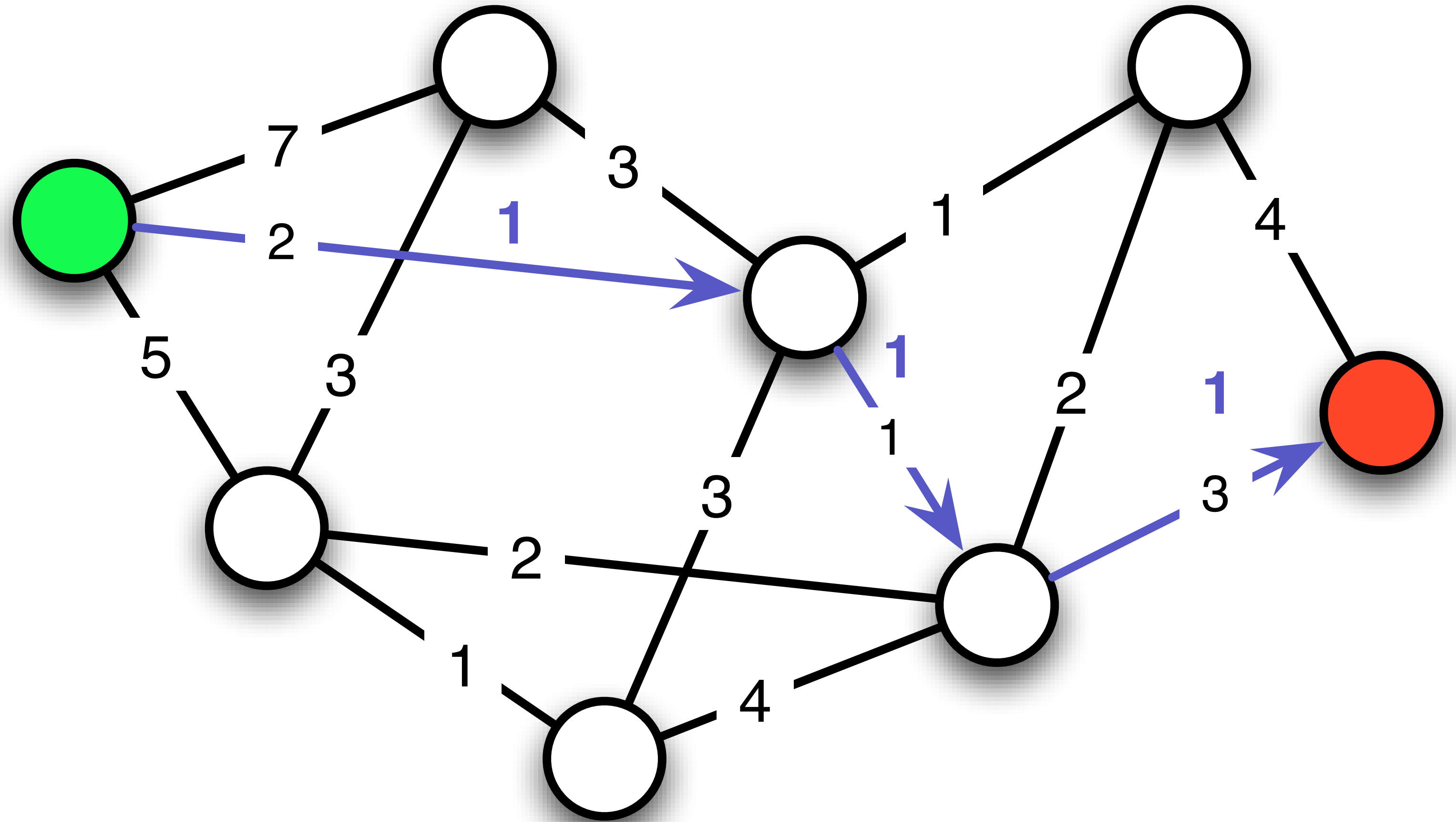
Edmunds-Karp

- **Search path for Ford-Fulkerson algorithm**
- **Choose the shortest augmenting path**
  - Computation by breadth-first-search
- **leads to run-time $O(|V|\ |E|^2)$**
  - whereas Ford-Fulkerson could have exponential run-time

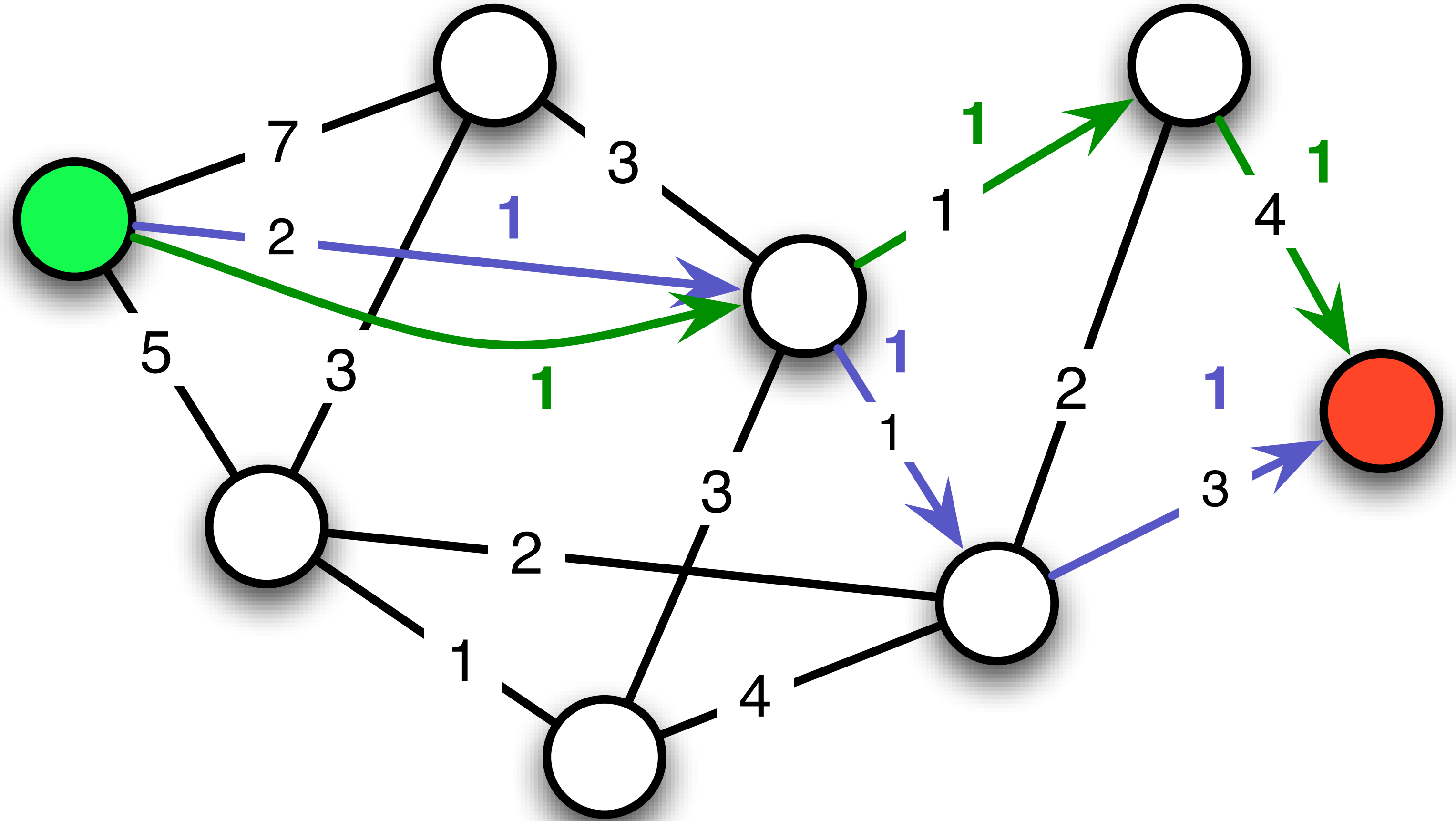
Example



Example



Example



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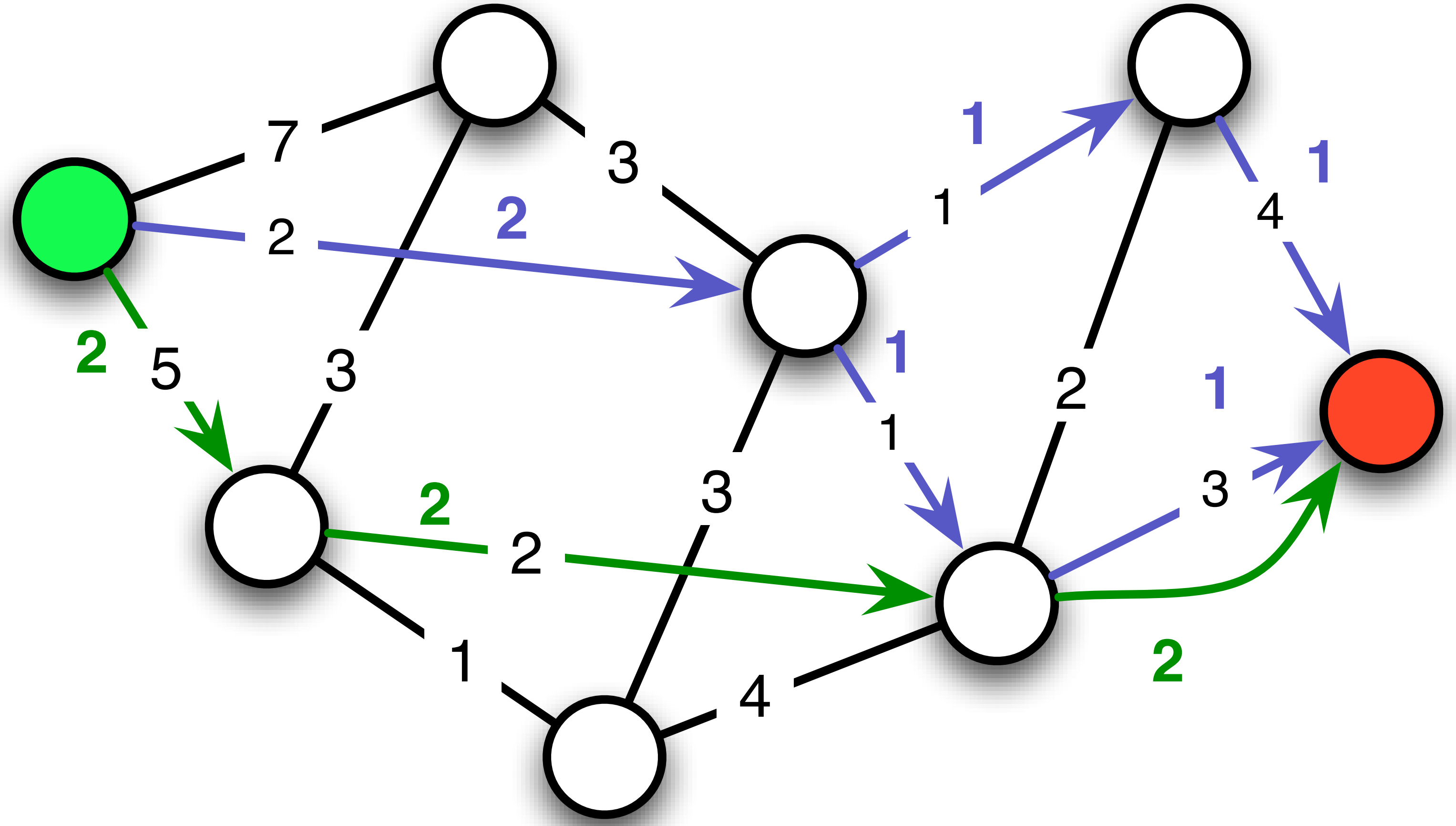

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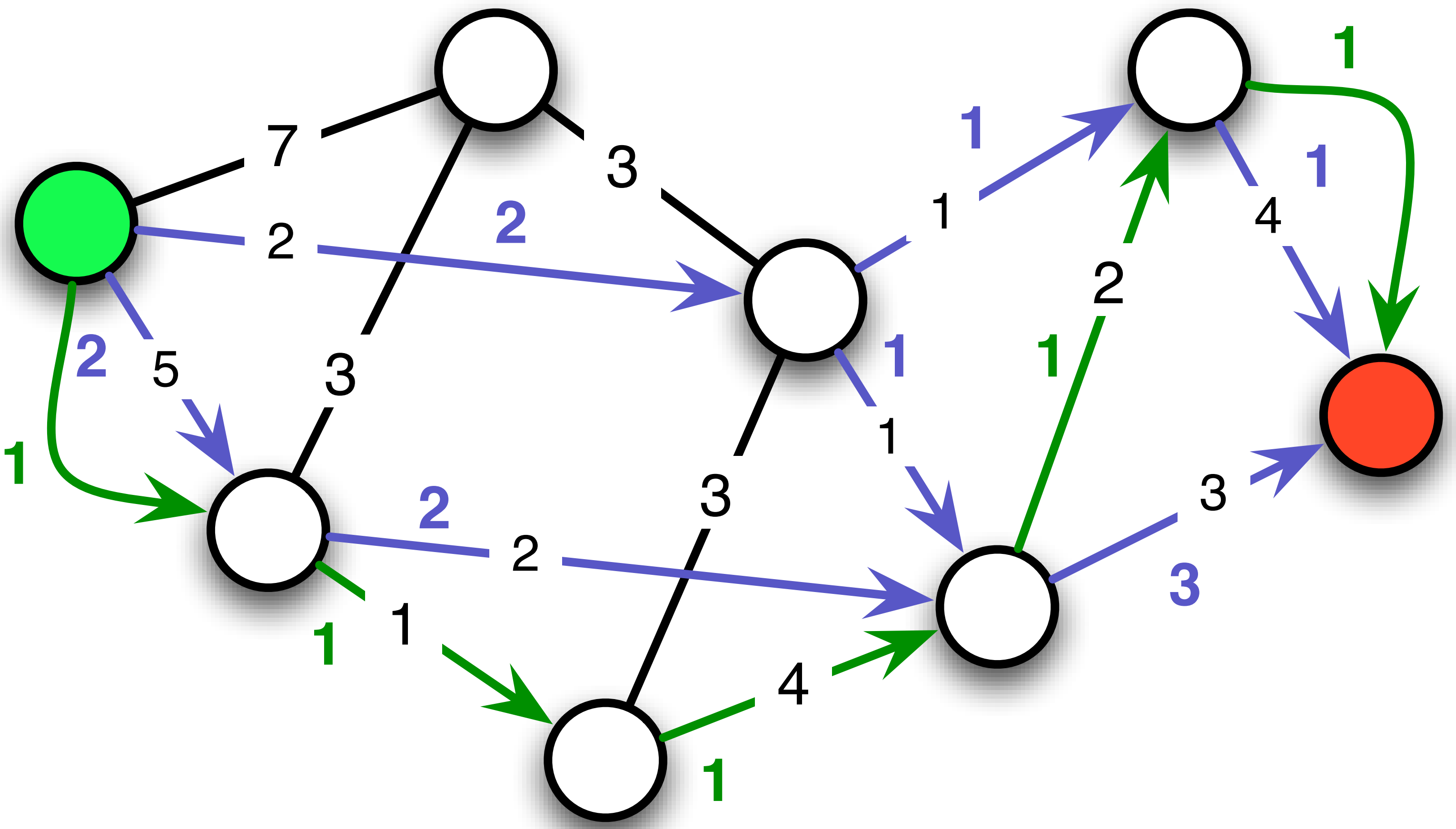

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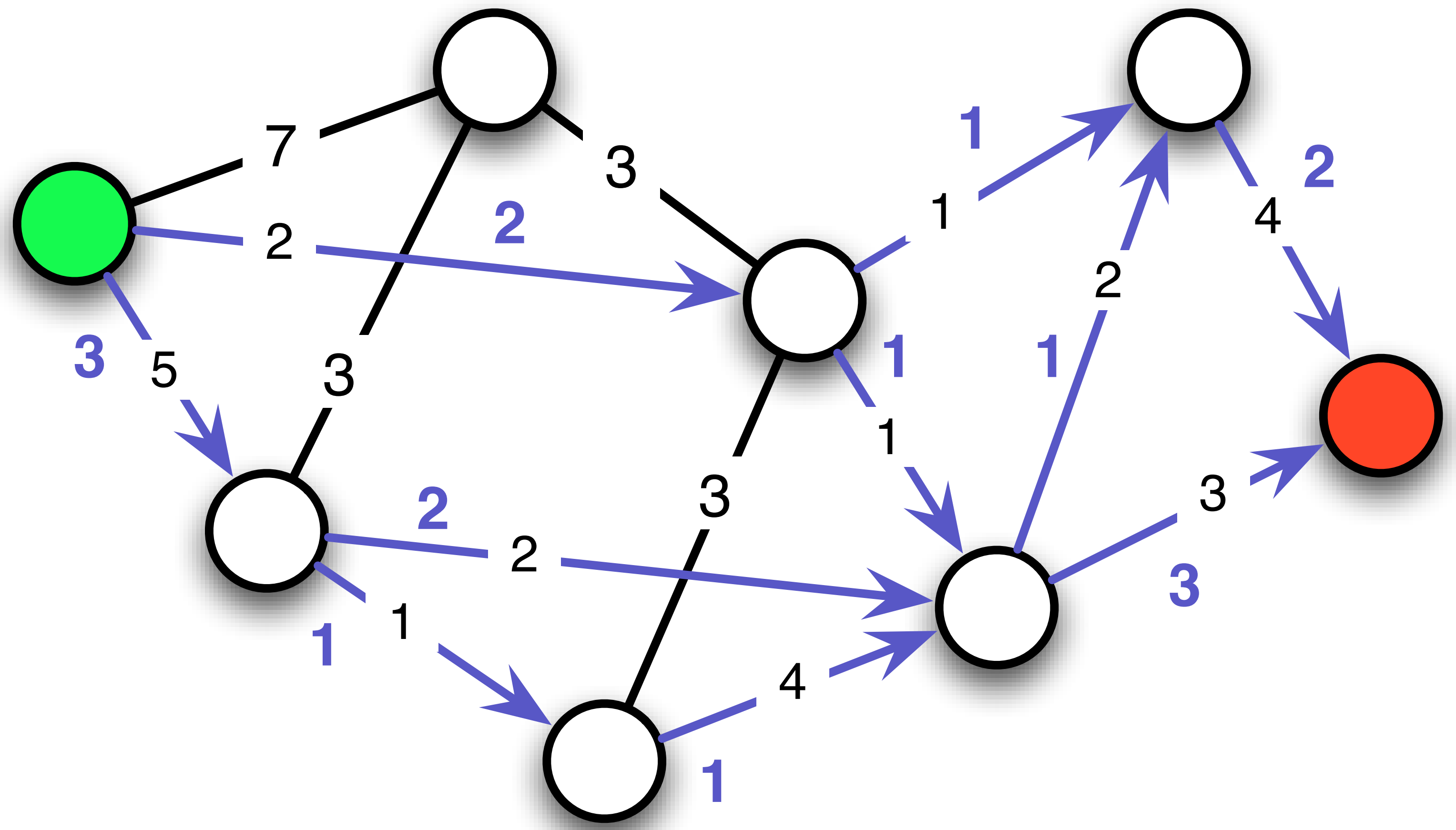

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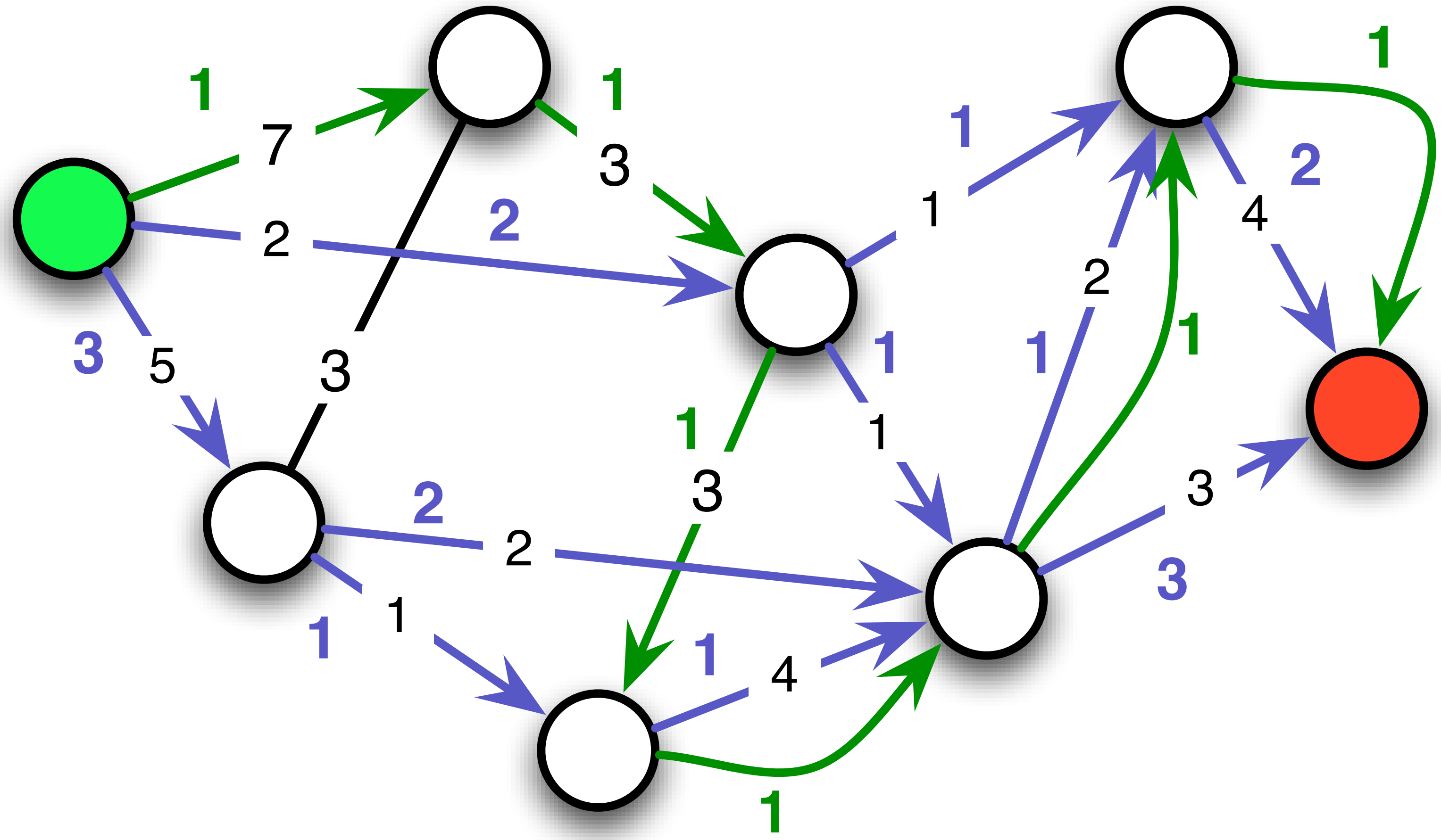

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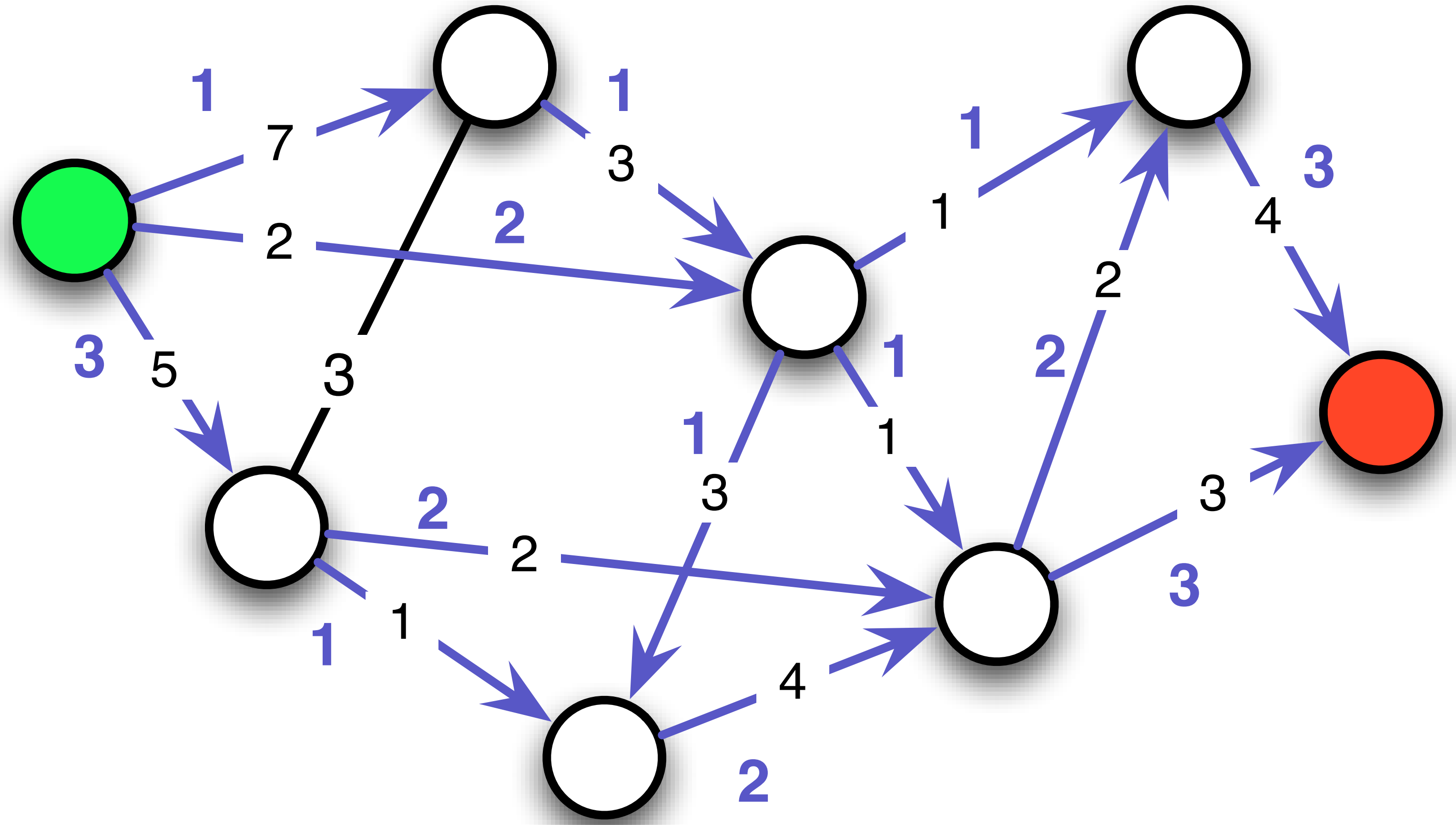
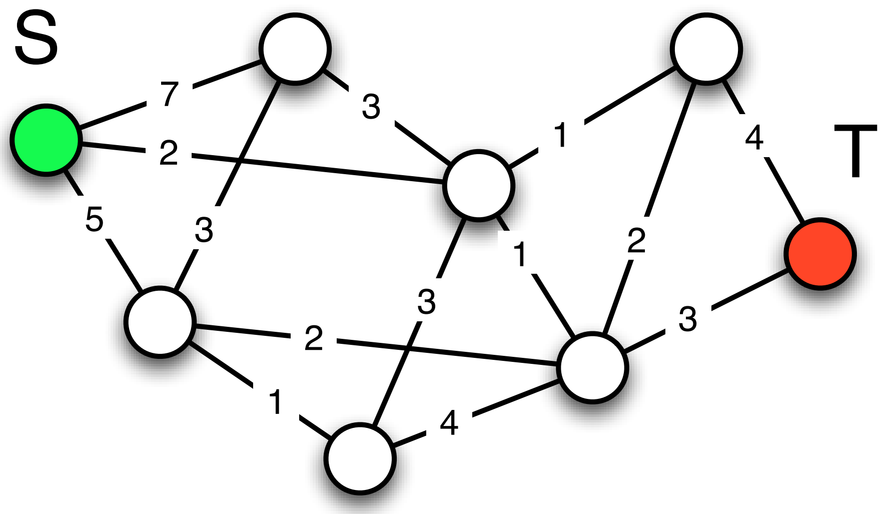
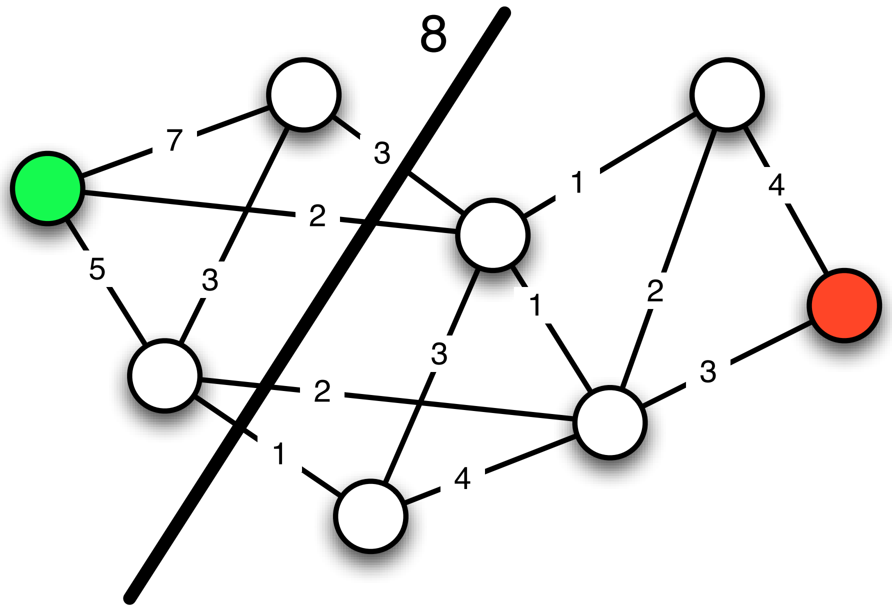


Minimum Cut in Networks

- **Motivation**
  - Find bottleneck in networks
- **Definition**
  - Min Cut problem
  - Given
    - graph G=(V,E)
    - capacity function w: E → R+0,
    - sources S and targets T
  - Find minimum cut between S and T
- **A cut C is a set of edges**
  - such that every path from a node of S to a node of T, contains an edge of C
- **The size of a cut is**

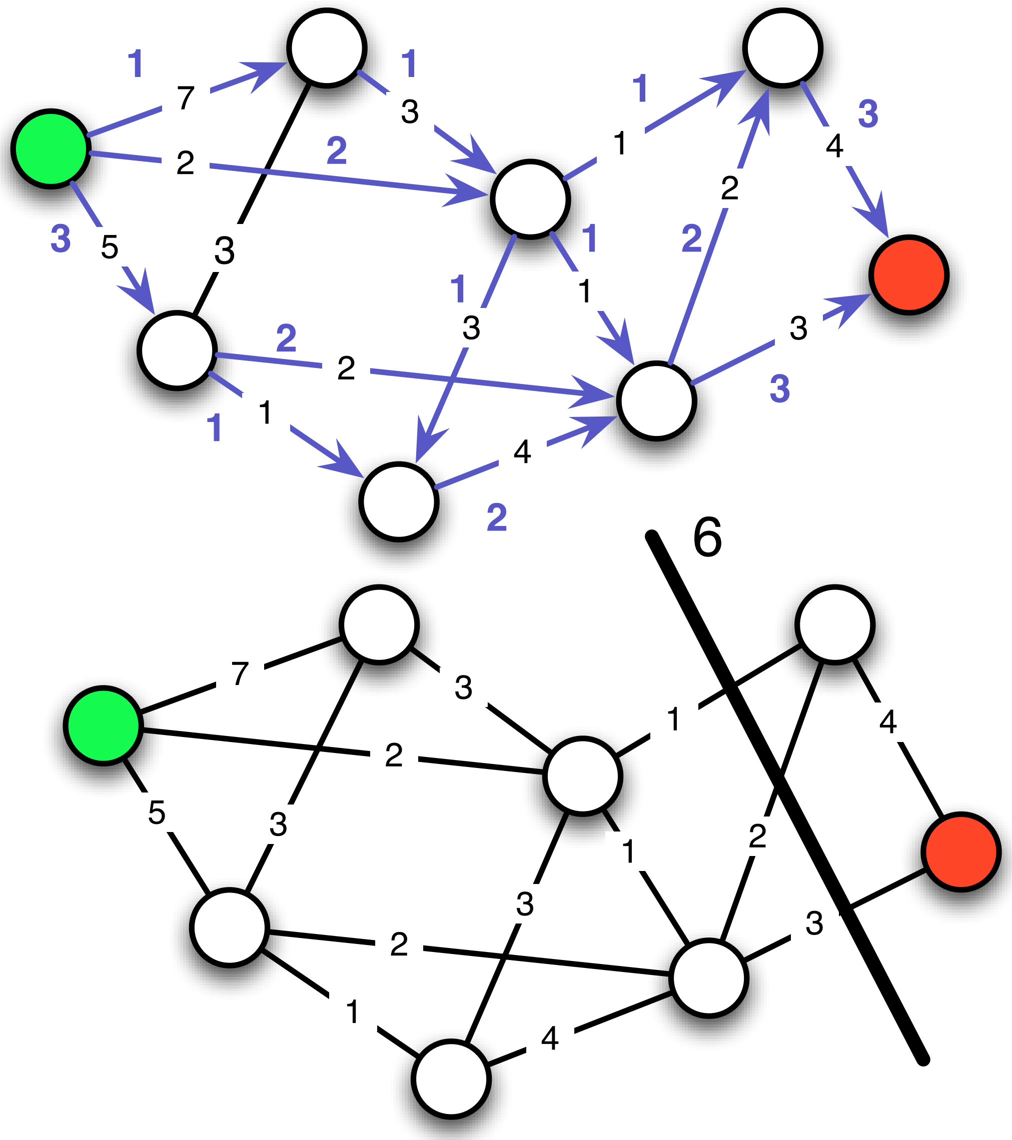
$$\sum_{e \in C} w(e)$$





Min-Cut-Max-Flow Theorem

- **Theorem**
  - The minimum cut equals the maximum flow
- **Algorithms for minimum cut**
  - can be obtained from the maximum flow algorithms



Multi-Commodity Flow Problem

- **Motivation**
  - theoretical model for point to point communication
- **Definition**
  - Multi-commodity flow problem
  - given
    - a graph G=(V,E)
    - a capacity function w: E → R+0,
    - commodities K_1, .., K_k:
      - $K_i=(s_i,t_i,d_i)$ with
      - s_i: source node
      - t_i: target node
      - d_i: demand
- **Find flows $f_1,f_2,\ldots,f_k$ for all commodities such that**
  - capacities

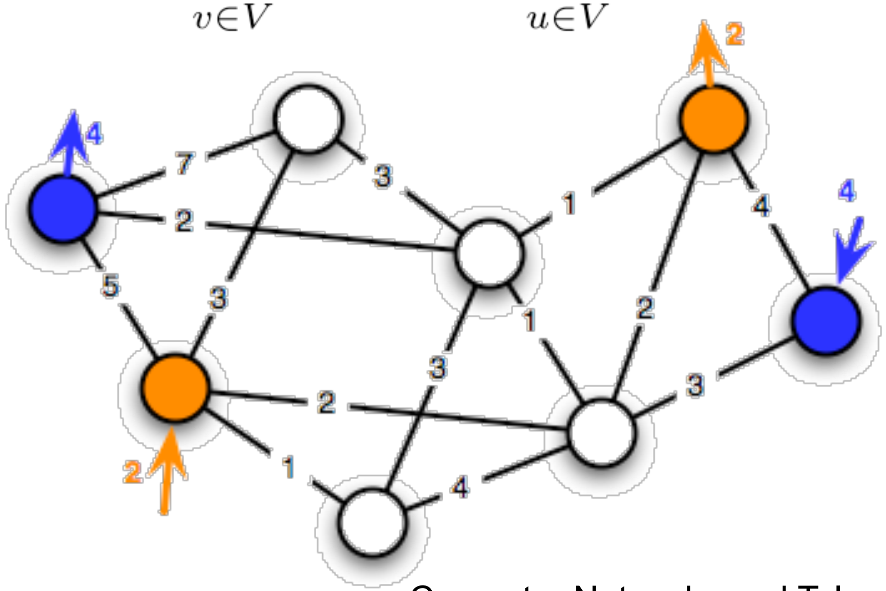
$$\sum_{i=1}^{k} f_i(u,v) \leq w(u,v)$$

  - flow property

$$\forall v \notin \{s_i,t_i\} \ : \ \sum_{u \in V} f_i(u,v) = \sum_{u \in V} f_i(v,u)$$

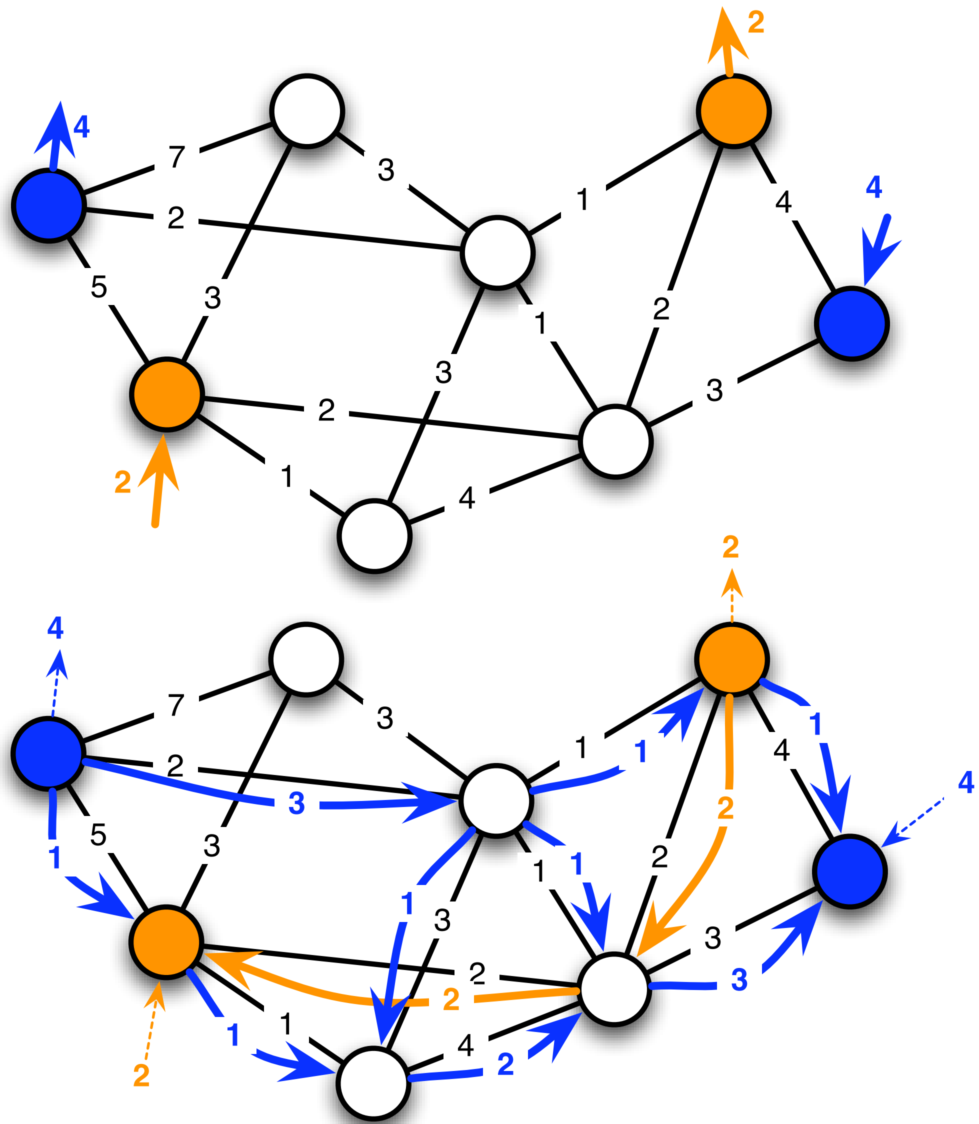
  - demand

$$\sum_{v \in V} f_i(s_i,v) = \sum_{u \in V} f_i(u,t_i) = d_i$$



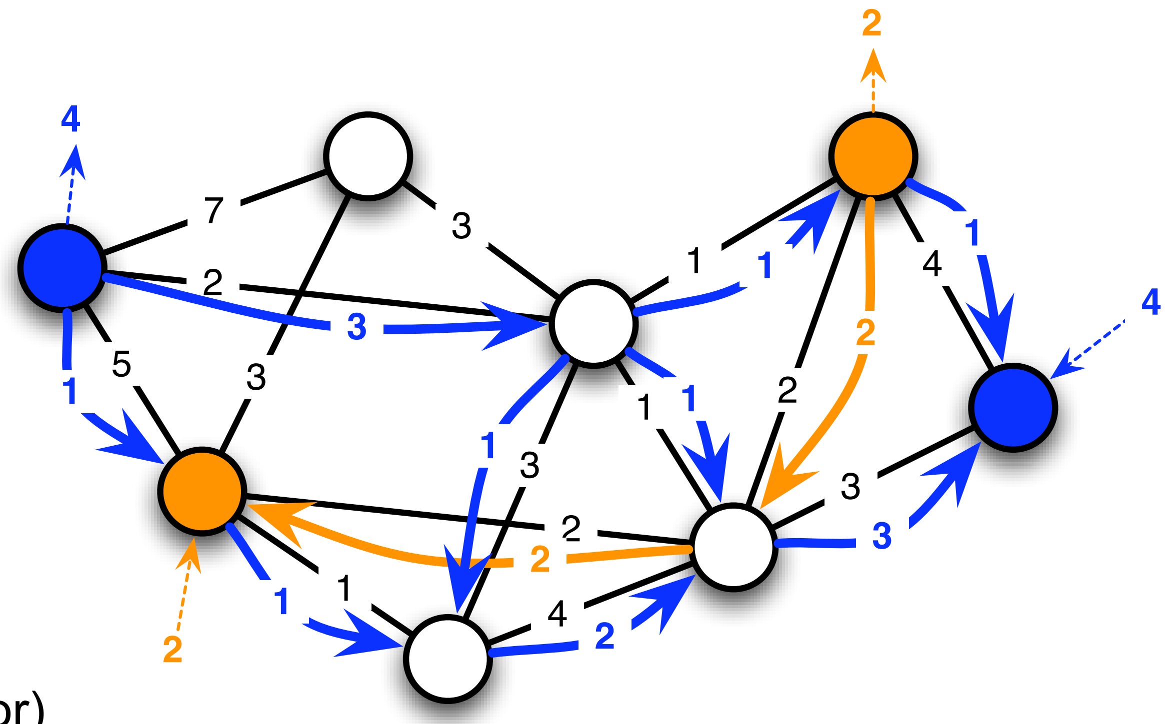
Solving the Multi-Commodity Flow Problem

- **Multi-Commodity Flow Problem**
- **Optimize**
  - sum of all flows or
  - maximize the worst ratio between commodity and the demand
- **Problem can be solved in polynomial time**
  - for real numbers
  - using linear programming



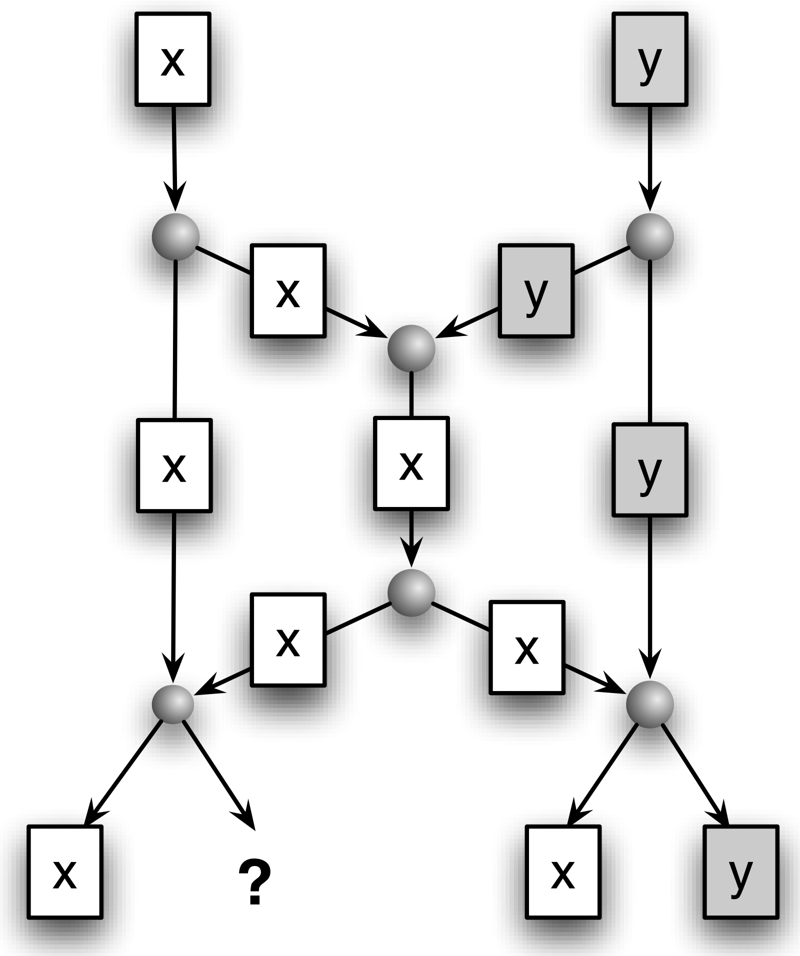
Complexity of the Multi Commodity Flow Problem

- **Problem is NP-complete**
  - for integers
    - e.g. packets
  - even for two commodities
    - Shai, Itai, Even, 1976
- **Polynomial solution**
  - with respect to the number of paths between sources and targets
- **Approximation**
  - good central and distributed approximation algorithms exist (polylogarithmic approximation factor)
- **Weaker forms of the Min-Cut-Max-Flow-Theorems exist**



Network Coding

- **R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung**
  - *Network Information Flow*, (IEEE Transactions on Information Theory, IT-46, pp. 1204-1216, 2000)
- **Example**
  - Bits x and y are to be transfered
  - Each edge carries only a bit
  - If bits are transfered as is
    - then both x and y cannot be received either on the left or right side



Network Coding

- **Example**
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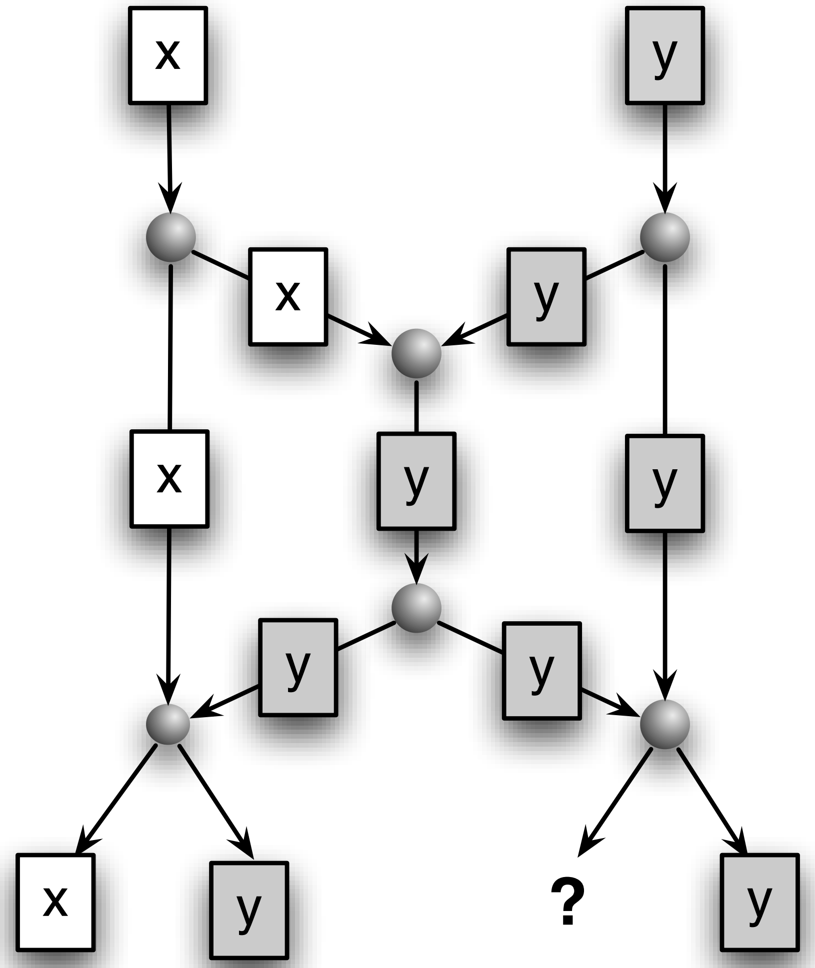
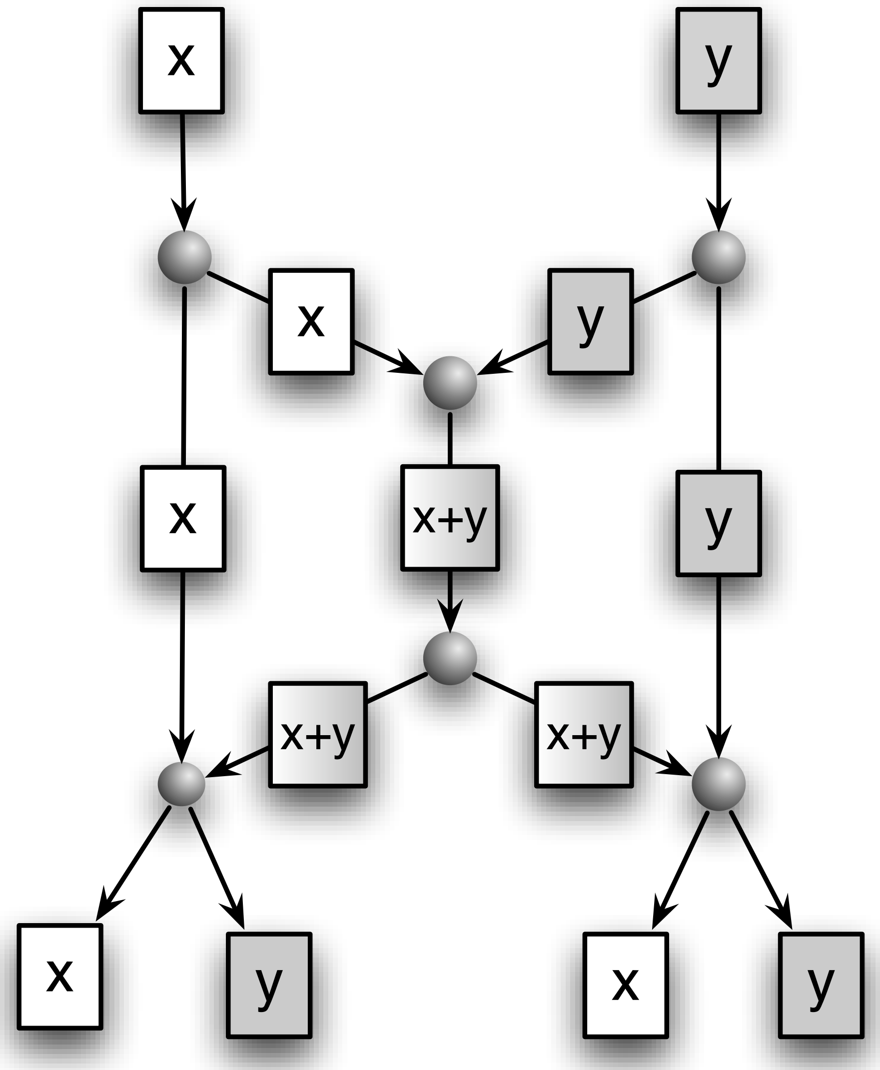


Network Coding

- **Solution**
  - Transfer Xor A+B on the middle edge



Network Coding and Flow

- **Theorem [Ahlswede et al.]**
  - For each graph there exists a network code such that each sink can receive as many bits as the maximum flow allows for each sink.

Linear Codes for Network Coding

- **Koetter, Médard**
  - Beyond Routing: An Algebraic Approach to Network Coding
- **Task**
  - Efficiently compute the network code
- **Solution**
  - Linear codes can always solve network coding
- **Practical Network Coding**
  - With high probability even random linear combinations suffice

Application Areas

- **Satellite Communication**
  - Preliminary work was published there
- **Peer-to-Peer networks**
  - Better information flow better than previous protocols
  - But too inefficient to displace prevalent protocols, e.g. Bittorrent
- **WLAN**
  - Xor in the Air, COPE
    - Simple network code improves flow
- **Ad-Hoc Networks, Wireless Sensor Networks, ...**

Coding and Decoding

- **Original message: $x_1, x_2, \ldots, x_m$**
- **Coding packet: $y_1, y_2, \ldots, y_m$**
- **Random variable r_{ij}**
- **Then:**

$$(r_{i1} r_{i2} \ldots r_{im}) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = y_i$$

$$\begin{pmatrix} r_{11} & \ldots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{m1} & \ldots & r_{mm} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

- **If the matrix (r_{ij}) is invertable**

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} r_{11} & \ldots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{m1} & \ldots & r_{mm} \end{pmatrix}^{-1} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Inverse of a Random Matrix

- **Theorem**
  - If the numbers of an m x m Matrix are chosen randomly from a finite field with b elements, then the matrix is invertable with proability of at least

$$1 - \sum_{i=1}^{m} \frac{1}{b^i}$$

- **Idea: Consider Galois-Field GF[2^k]**
  - Computation is efficient
  - Binary represenation of data straight-forward

Galois Field

- **$GF(2^w)$ = finite field with 2^w elements**
  - elements are binary strings of length w
  - $0 = 0^w$ neutral element of addition
  - $1 = 0^{w-1}1$ neutral element of multiplikation
- **u + v = bit-wise Xor of strings**
  - z.B. 0101 + 1100 = 1001
- **a b = product of polynomials modula a given irreducible polynomial and modulo 2**
  - i.e. $(a_{w-1} \ldots a_1\ a_0)\ (b_{w-1} \ldots b_1\ b_0) =$

$$((a_0 + a_1 x + \ldots + a_{w-1} x^{w-1})(b_0 + b_1 x + \ldots + b_{w-1} x^{w-1}) \bmod q(x)) \bmod 2)$$

Example: $GF(2^2)$

$q(x) = x^2+x+1$

Generator of GF(4)	Polynomial in GF(4)	Binary Represen-tation in GF(4)	Decimal Represen-tation
0	0	00	0
x^0	1	01	1
x^1	x	10	2
x^2	x+1	11	3

Example: GF(2^2)

+	0 = 00	1 = 01	2 = 10	3 = 11
0 =00	00	01	10	11
1 =01	01	00	11	10
2 =10	10	11	00	01
3 =11	11	10	01	00

Example: GF(2^2)

$q(x) = x^2+x+1$

*	0 = 0	1 = 1	2 = x	3 = x^2
0 = 0	0	0	0	0
1 = 1	0	1	x	x^2
2 = x	0	x	x^2	1
3 = x^2	0	x^2	1	x

Irreducible Polynomial

- **Irreducible polynomial cannot be factorized**
  - **Irreducible polynomial** $x^2+1 = (x+1)^2 \bmod 2$
- **Irreducible polynomials**
  - w=2: x^2+x+1
  - w=4: x^4+x+1
  - w=8: $x^8+x^4+x^3+x^2+1$
  - w=16: $x^{16}+x^{12}+x^3+x+1$
  - w=32: $x^{32}+x^{22}+x^2+x+1$
  - w=64: $x^{64}+x^4+x^3+x+1$

Fast Multiplication

- **Power law**
  - Consider $\{2^0, 2^1, 2^2, ...\}$
  - $= \{x^0, x^1, x^2, x^3, ...$
  - = exp(0), exp(1), ...
- **exp(x+y) = exp(x) exp(y)**
- **Inverse function: log(exp(x)) = x**
  - $\log(x \cdot y) = \log(x) + \log(y)$
- **x y = exp(log(x) + log(y))**
  - Caution: in the exponent standard addition
- **Tables store exponential function and logarithm**

Example: GF(16)

$q(x) = x^4+x+1$

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
exp(x)	1	x	x^2	x^3	$1+x$	$x+x^2$	x^2+x^3	$1+x+x^3$	$1+x^2$	$x+x^3$	$1+x+x^2$	$x+x^2+x^3$	$1+x+x^2+x^3$	$1+x^2+x^3$	$1+x^3$	1
exp(x)	1	2	4	8	3	6	12	11	5	10	7	14	15	13	9	1

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
log(x)	0	1	4	2	8	5	10	3	14	9	7	6	13	11	12

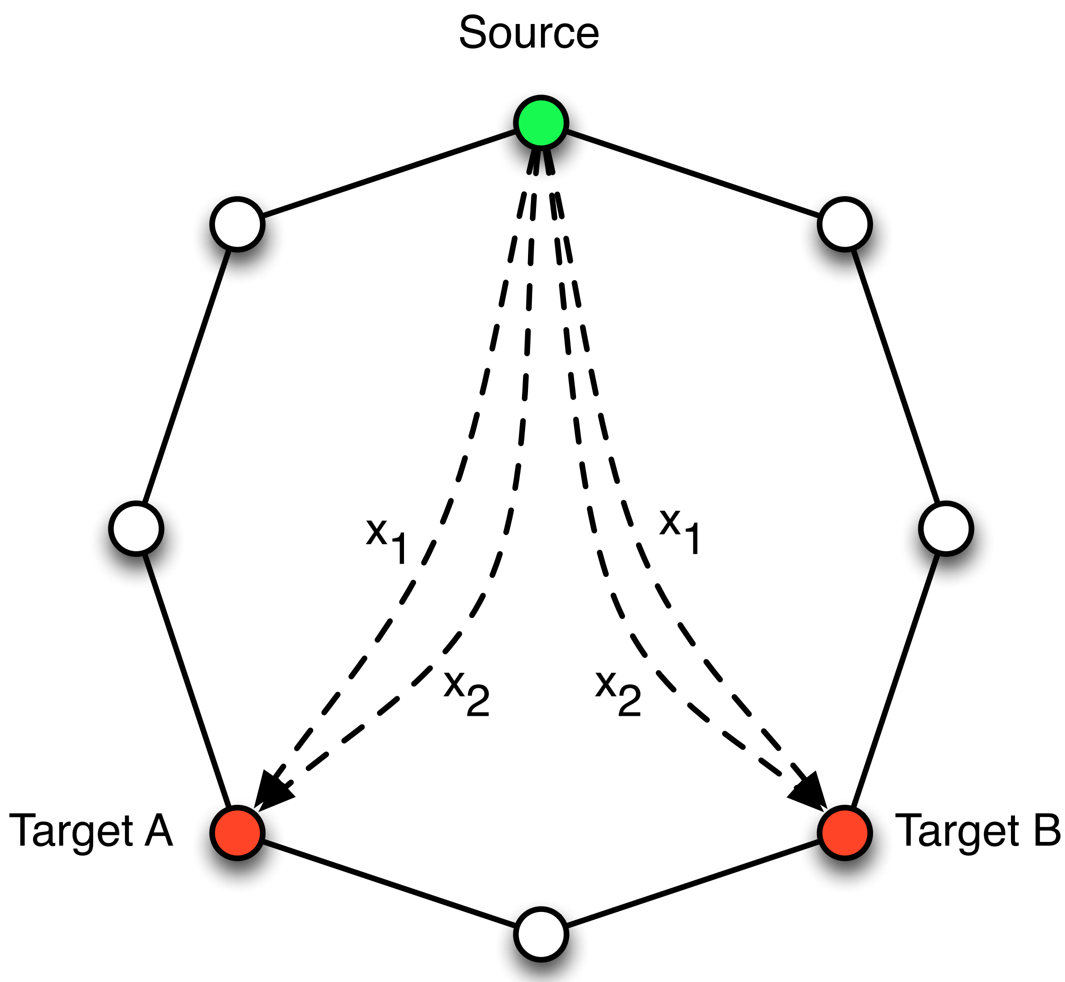
- $5 \cdot 12 = \exp(\log(5)+\log(12)) = \exp(8+6) = \exp(14) = 9$
- $7 \cdot 9 = \exp(\log(7)+\log(9)) = \exp(10+14) = \exp(24) = \exp(24-15) = \exp(9) = 10$

Special Case GF[2]

- **Network Coding in GF[2]**
  - Boolean Algebra
    - x + y = x XOR y
    - x   y = x AND y
- **Example**
  - Xor in the Air
  - Multicasting in Ad-Hoc Networks
- **Disadvantage**
  - Full potential of network coding is unused
- **Advantage**
  - Transparent, intuitiv and very efficient

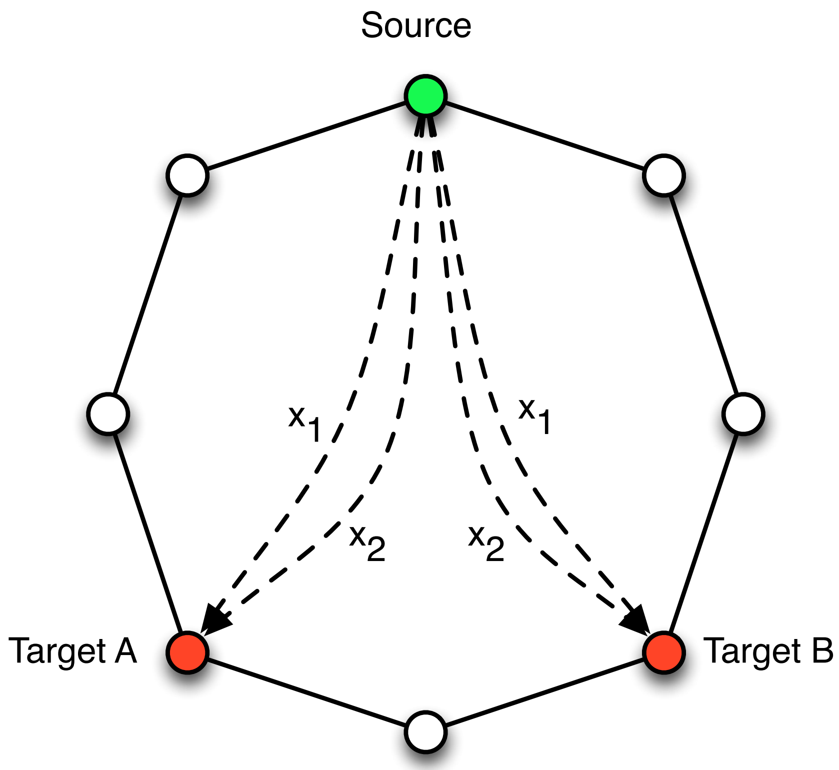
Multicasting in Ad Hoc Networks

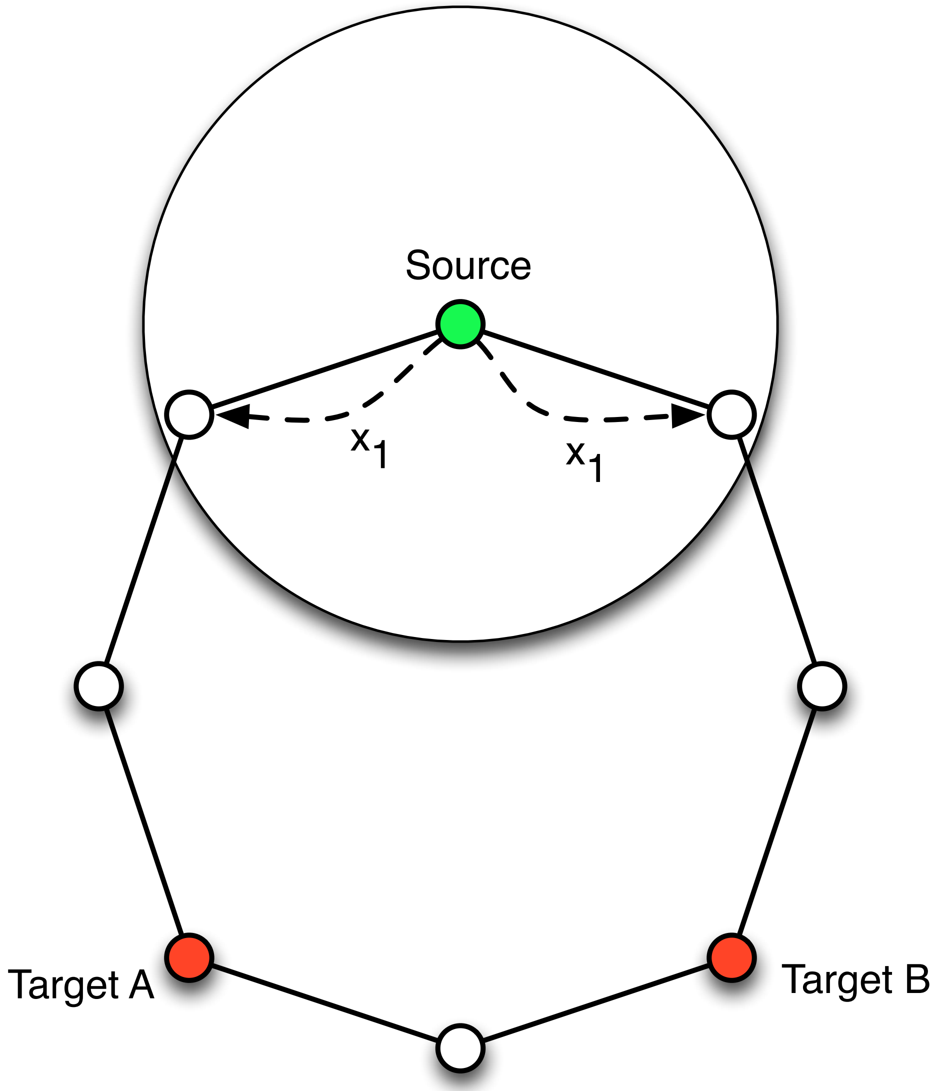
- **Wu, Chou, Sun-Yuan,**
  - Minimum-Energy Multicast in Mobile Ad hoc Networks using Network Coding, 2006
- **Multicast**
  - Distribute message from one node to a given set of nodes
- **Cost measure**
  - Each one-hop broadcast costs an energy unit



Beispiel

- **Traditionally,**
  - it costs 5 energy units for a multicast message





Example

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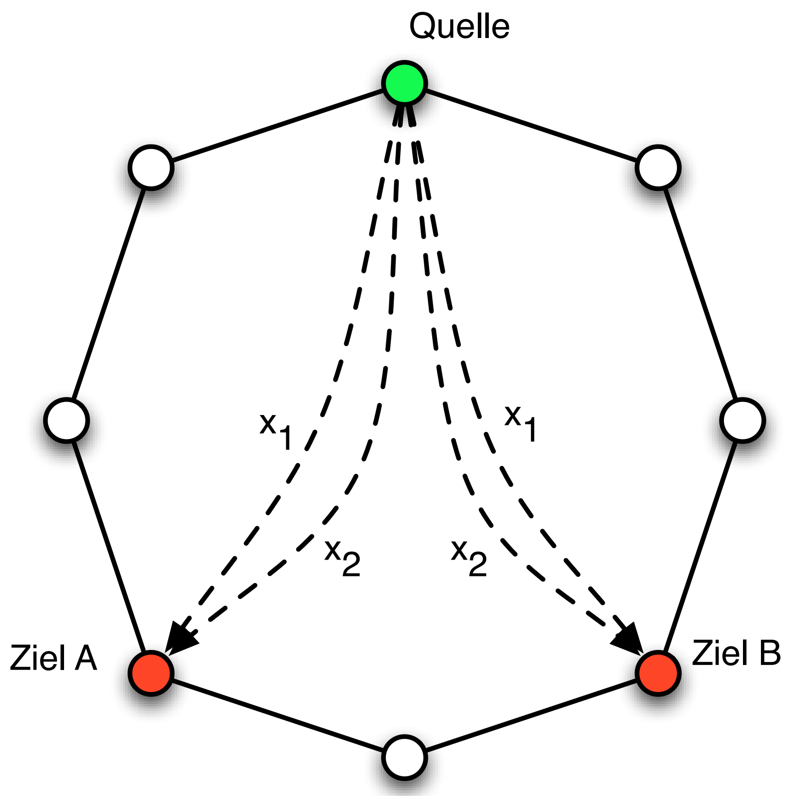
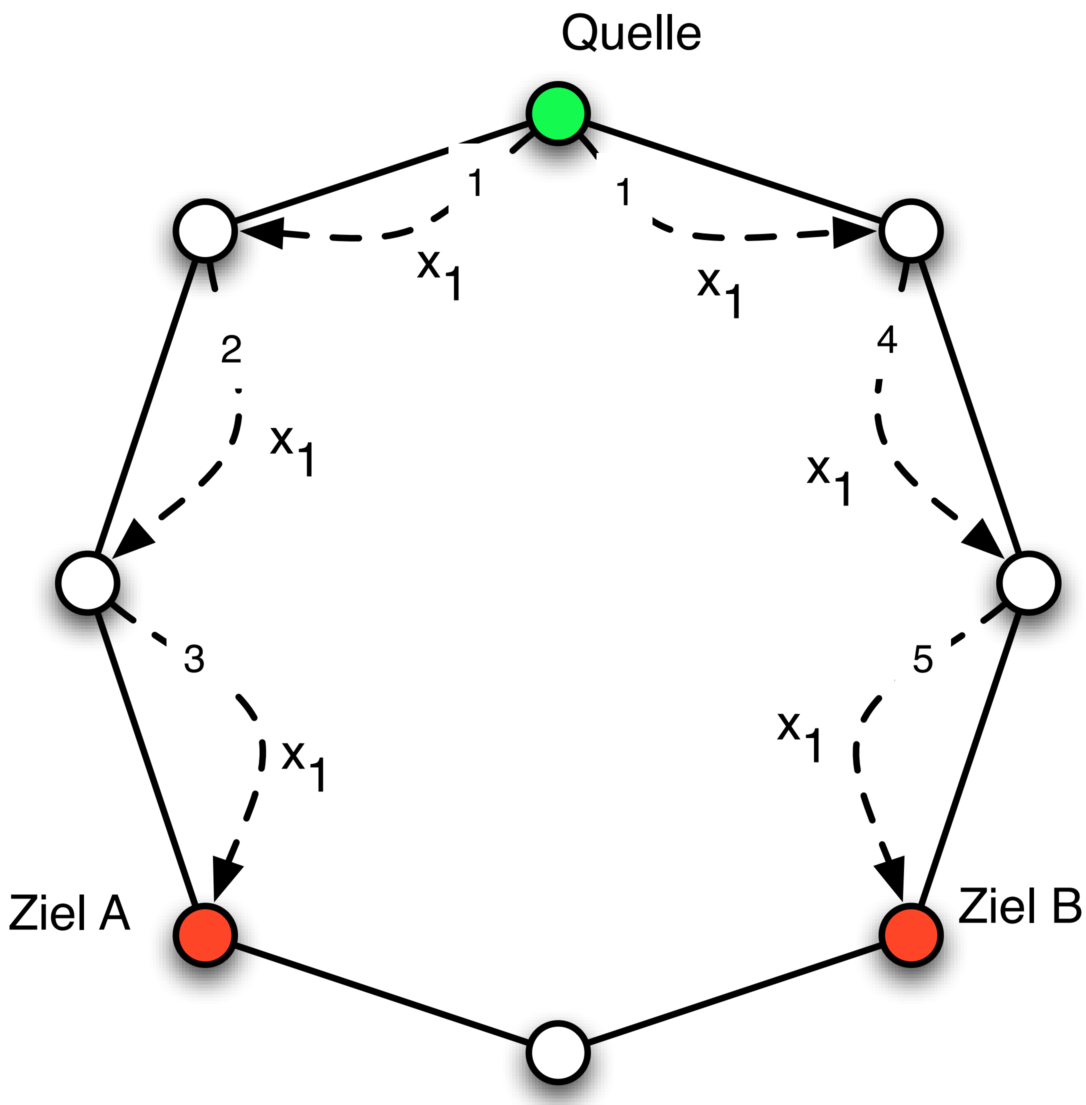
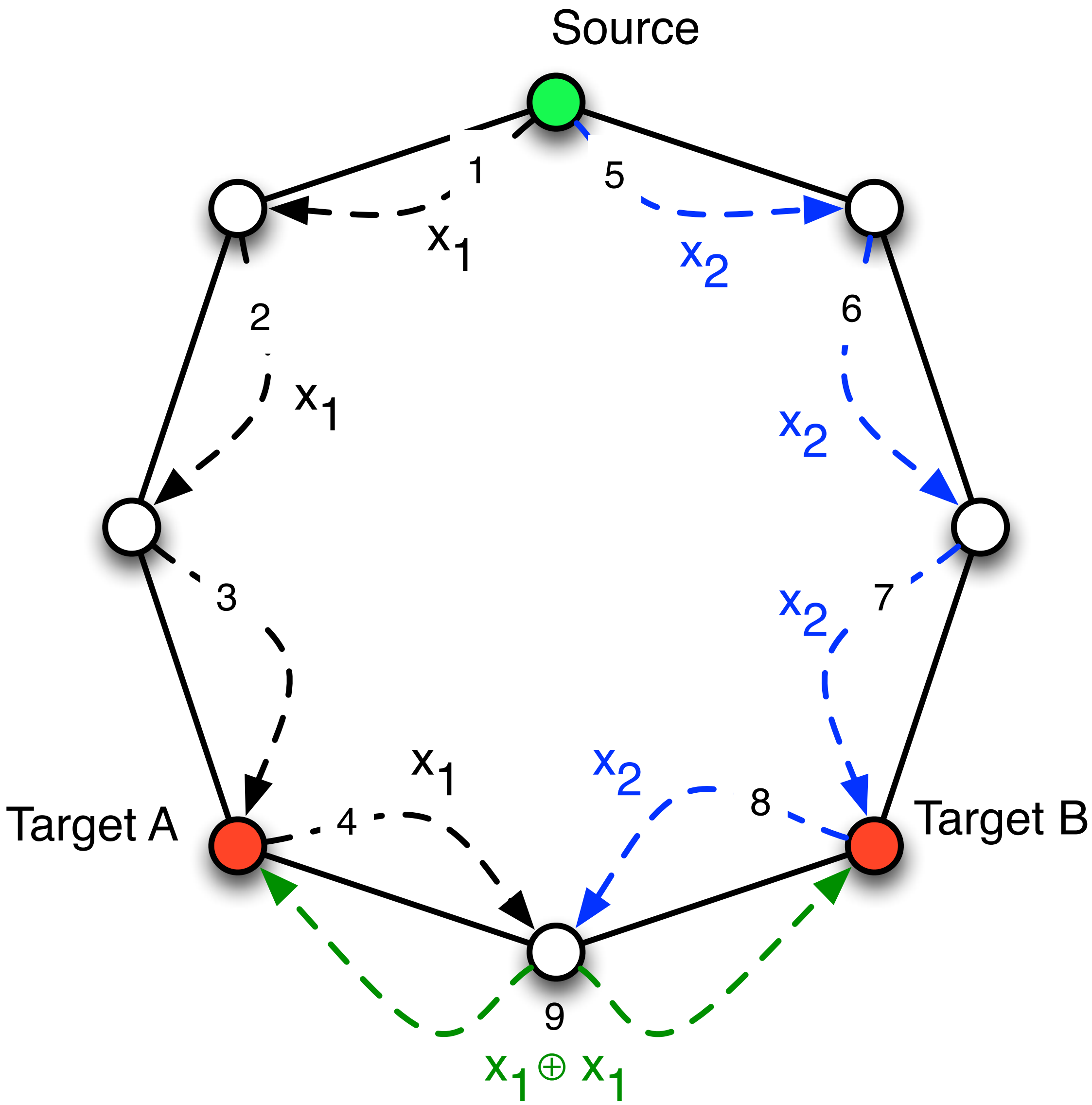




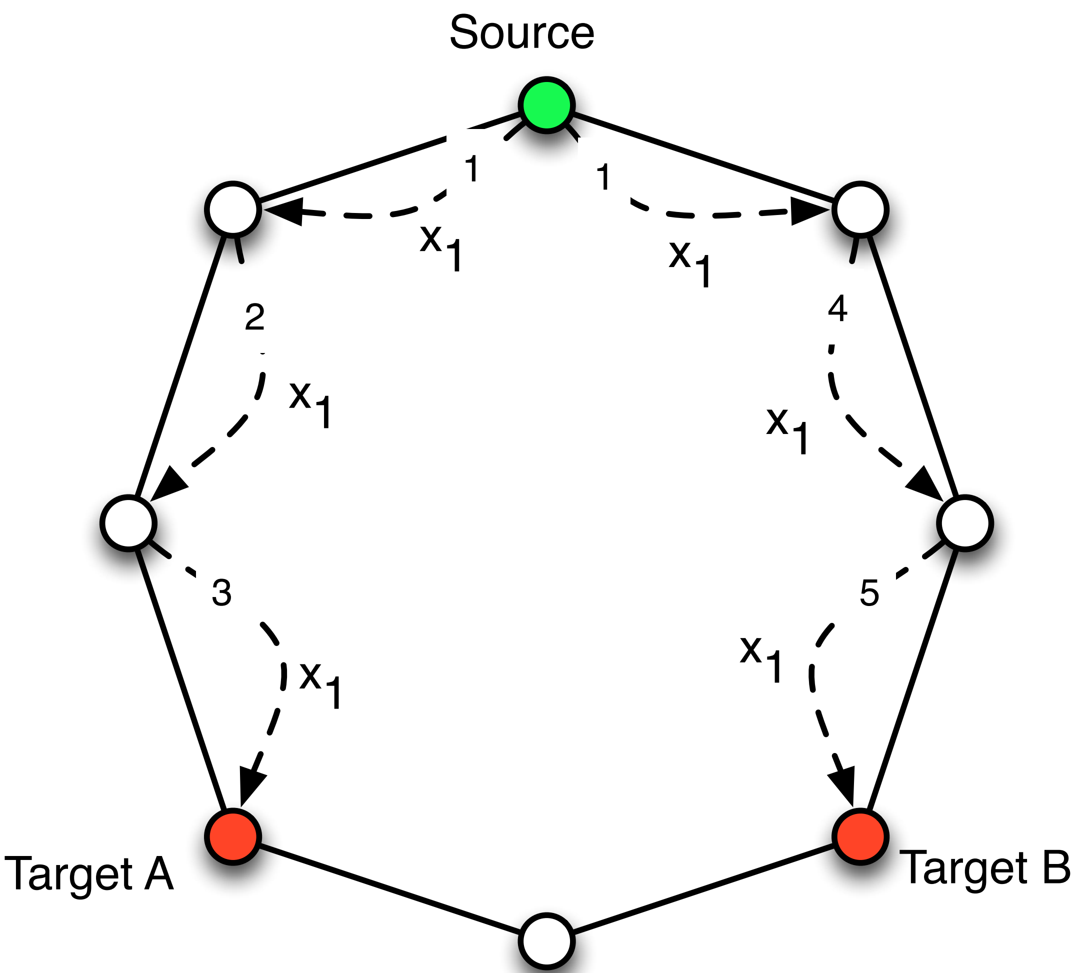
Example

- **Network coding**
  - 9 energy units for 2 messages
  - Average of 4.5
- **Without network coding**
  - 5 units for one multicast message

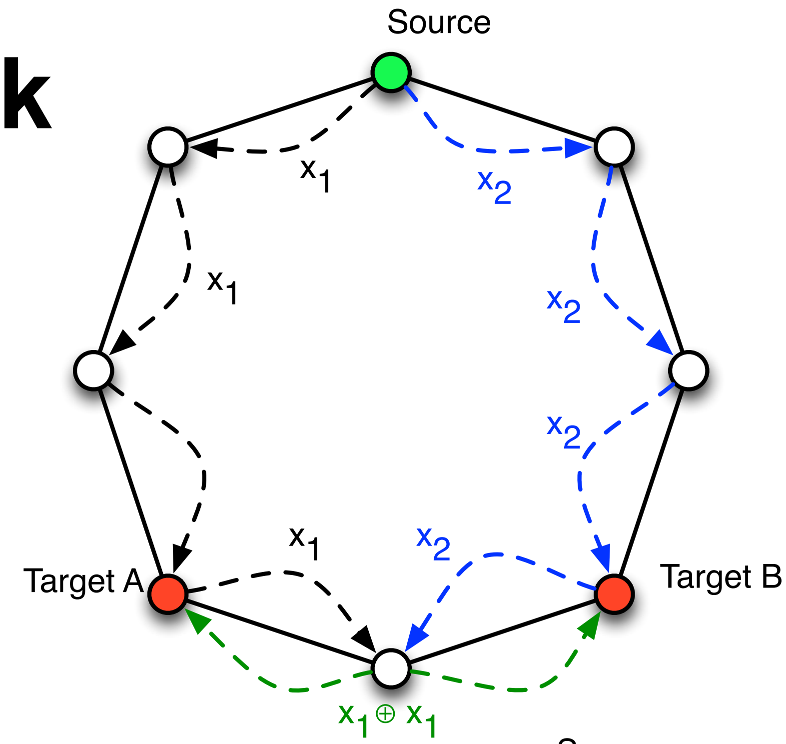


Multicasting in Ad Hoc Networks

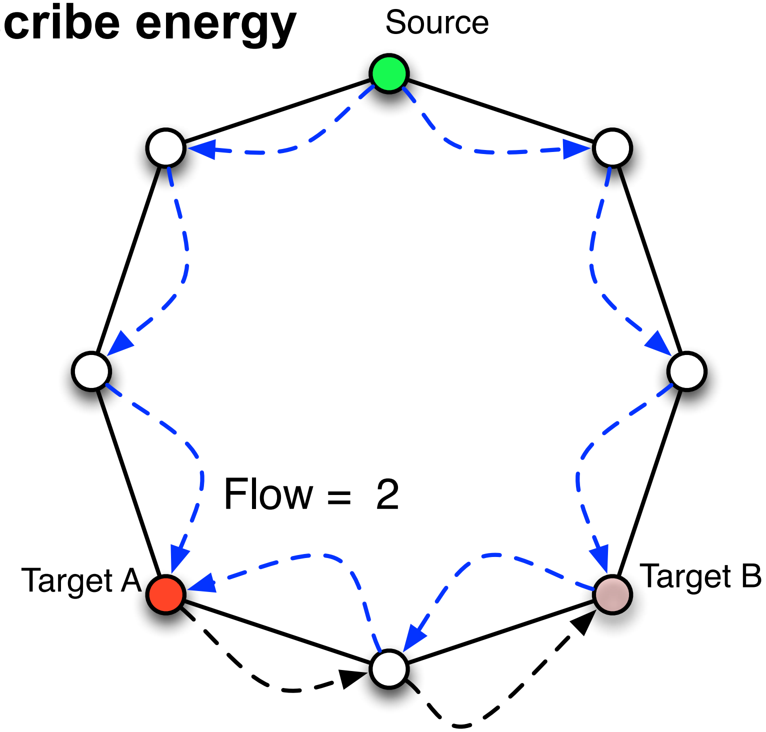
- **Solution of the minimal energy multicasting problem without network coding is NP-hard**
  - Less than constant factor approximation is NP-hard
  - Requires calculation of the discrete Steiner tree

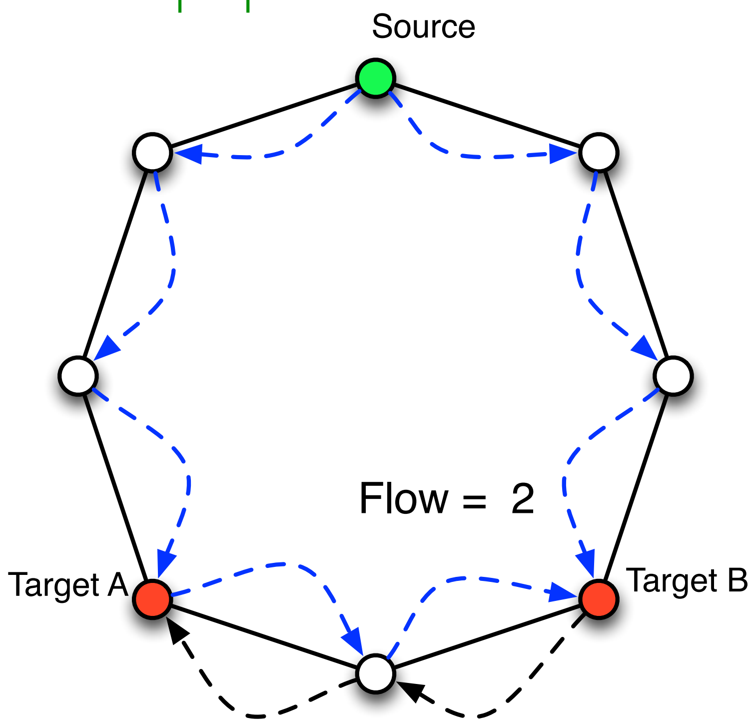


Condition for Network Coding



- **Messages allow flow of the size of the desired number of messages**
  - from the sources to each individual sink
- **If such flows are guaranteed, network coding can be applied**
- **Size of the flows describe energy consumption**

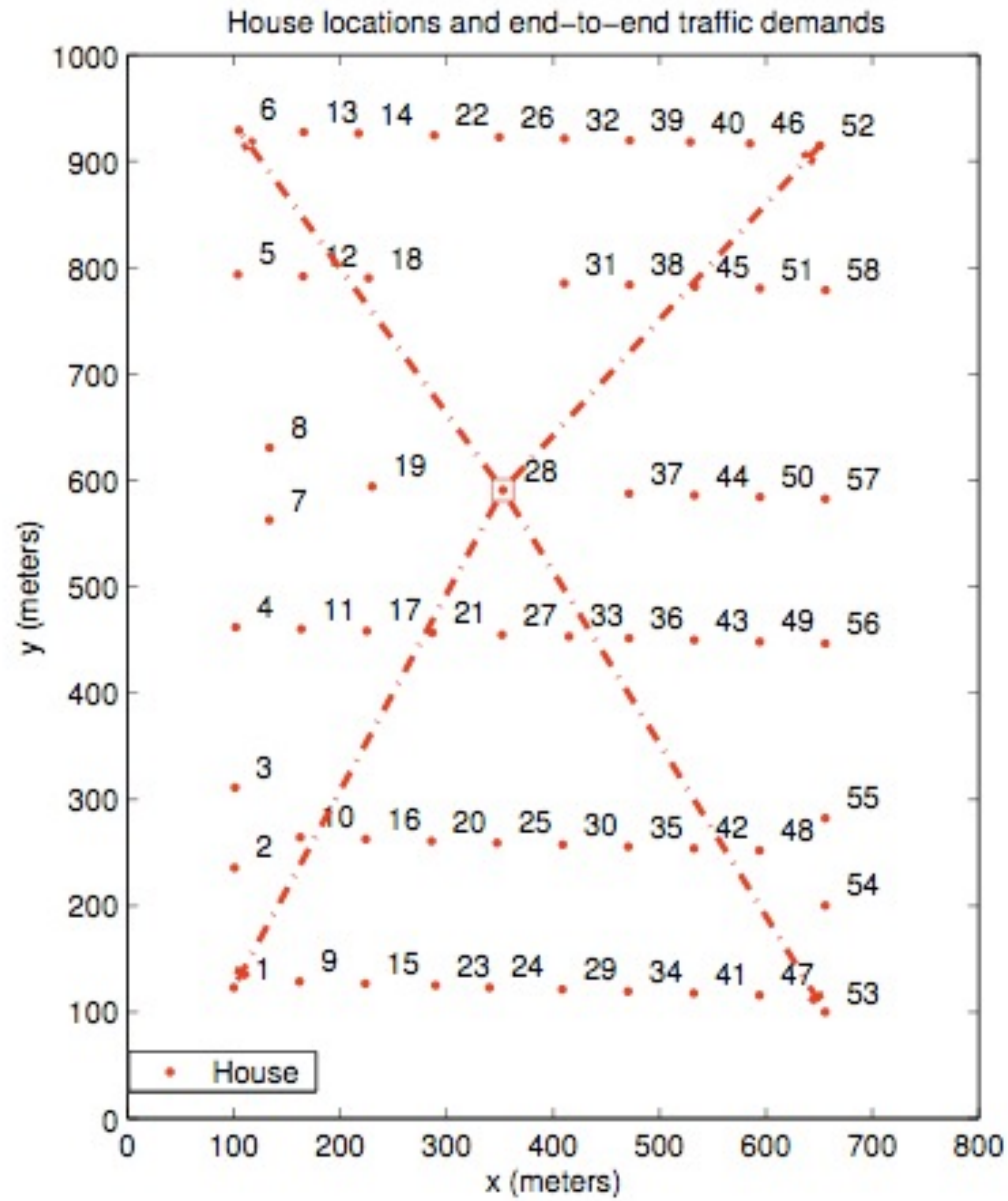




Computational Complexity

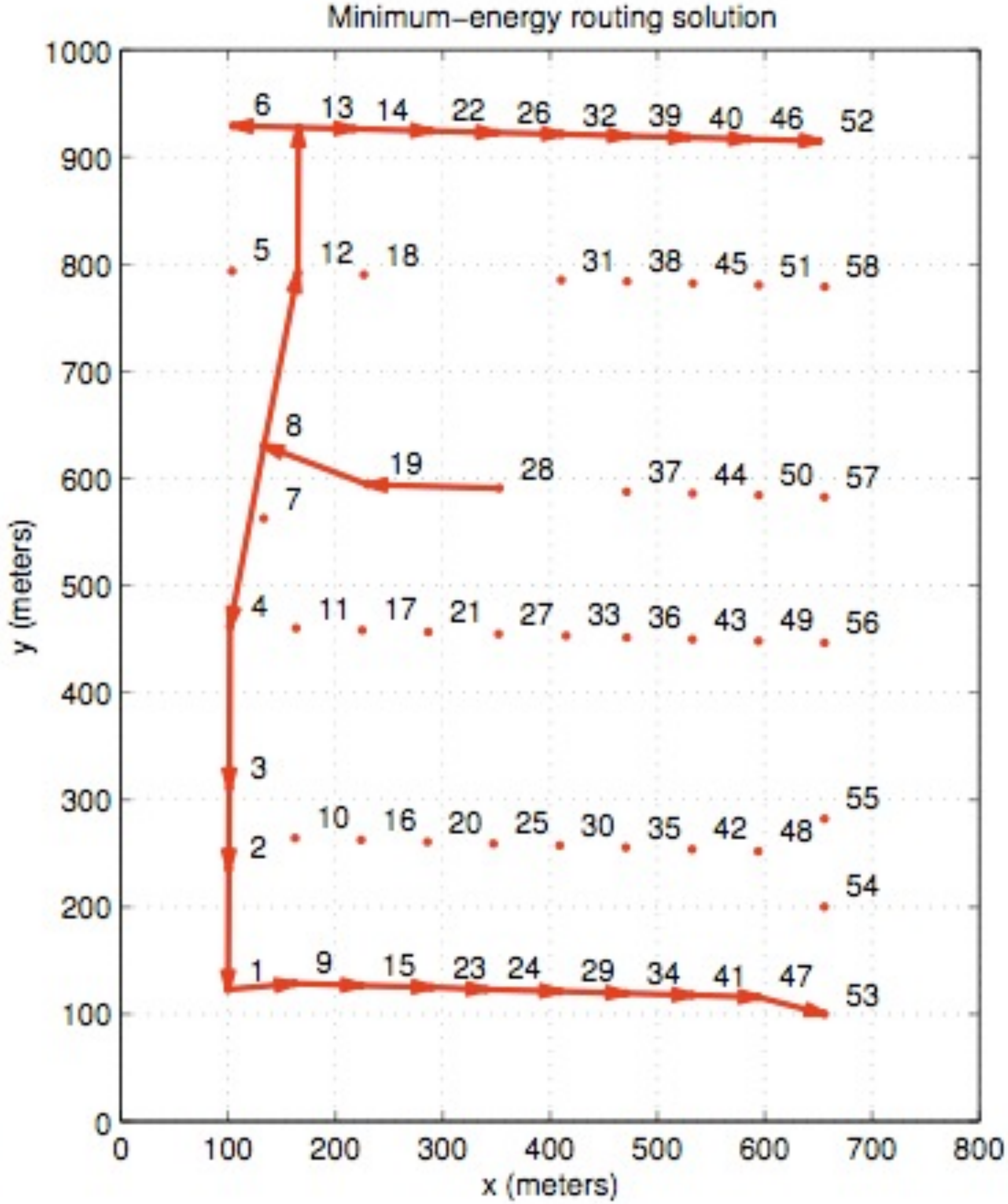
- **Algorithm**
  - Collect all available link information
  - Formulate as linear program
  - Approximation of the solution
- **With the help of network coding, the maximum throughput can be approximated arbitrarily well in polynomial time**

Example Demand



Wu, Chou, Sun-Yuan,
Minimum-Energy Multicast in Mobile Ad hoc Networks using Network Coding, 2006

Example Multicasting with minimal Energy



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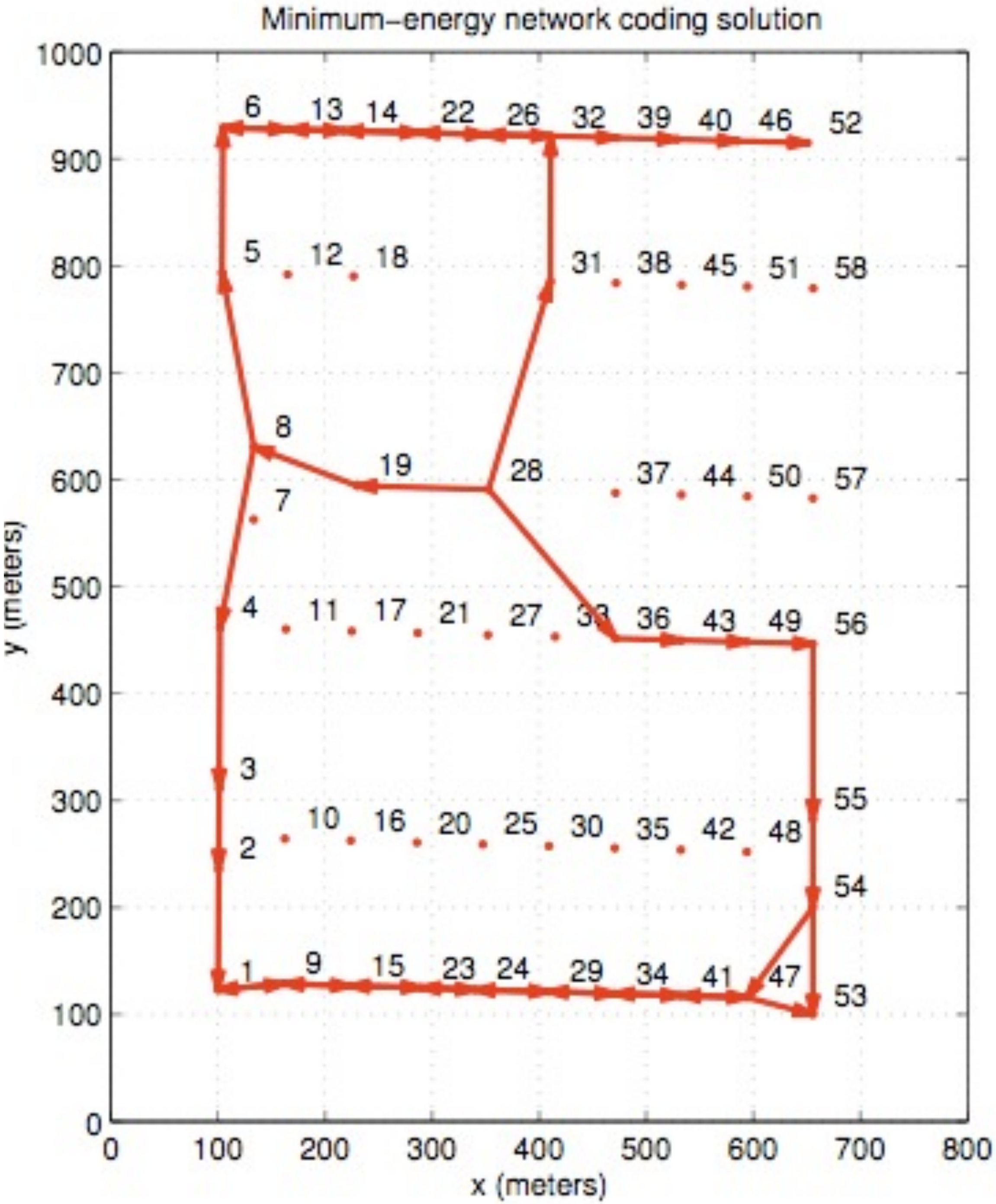
Multicasting with Network Coding



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Discussion

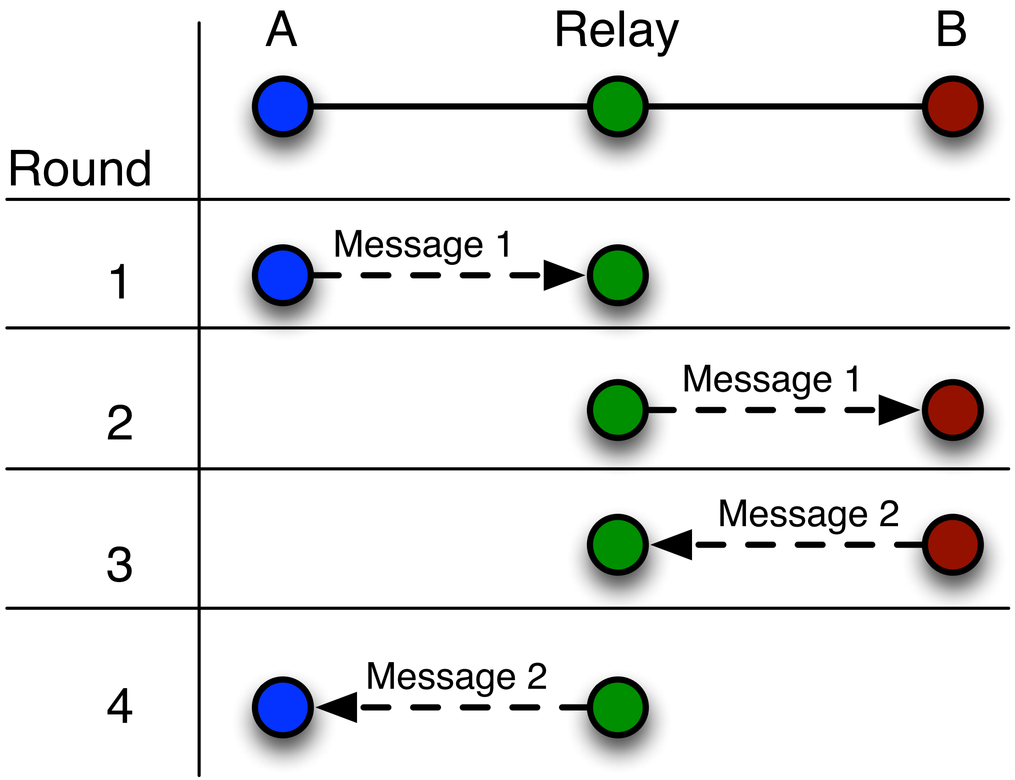
- **Options**
  - Energy model can customized
- **Limitations**
  - Network coding is not described
  - Central algorithm
  - Any change in the communication requires recalculation

Xors in the Air

- **Katti, Hu, Katabi, Médard, Crowcroft**
  - XORs in the Air: Practical Wireless Network Coding
- **Problem**
  - Maximize throughput in ad-hoc network
  - Multihop messages cause interference
- **Solution**
  - Uses only XORs of multiple messages
  - Local, opportunistic algorithm

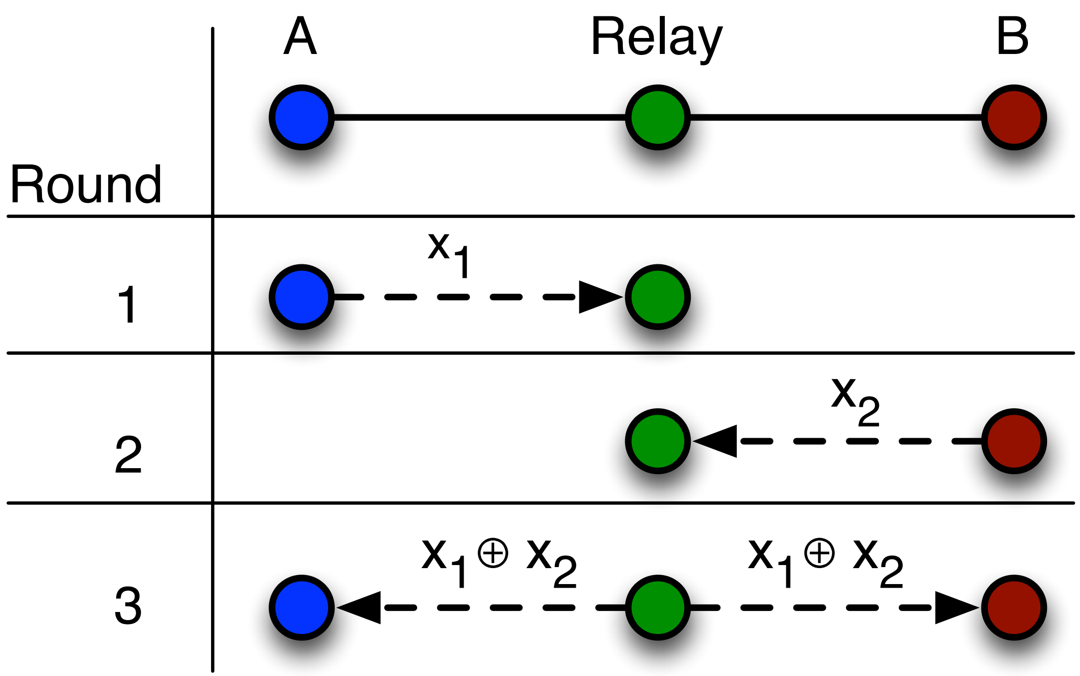
Xors in the Air

- **Problem**
  - Multihop messages cause interferences
- **Example**
  - Traditional: 4 messages to send
    - a message from A to B
    - and a message from B to A



Xors in the Air

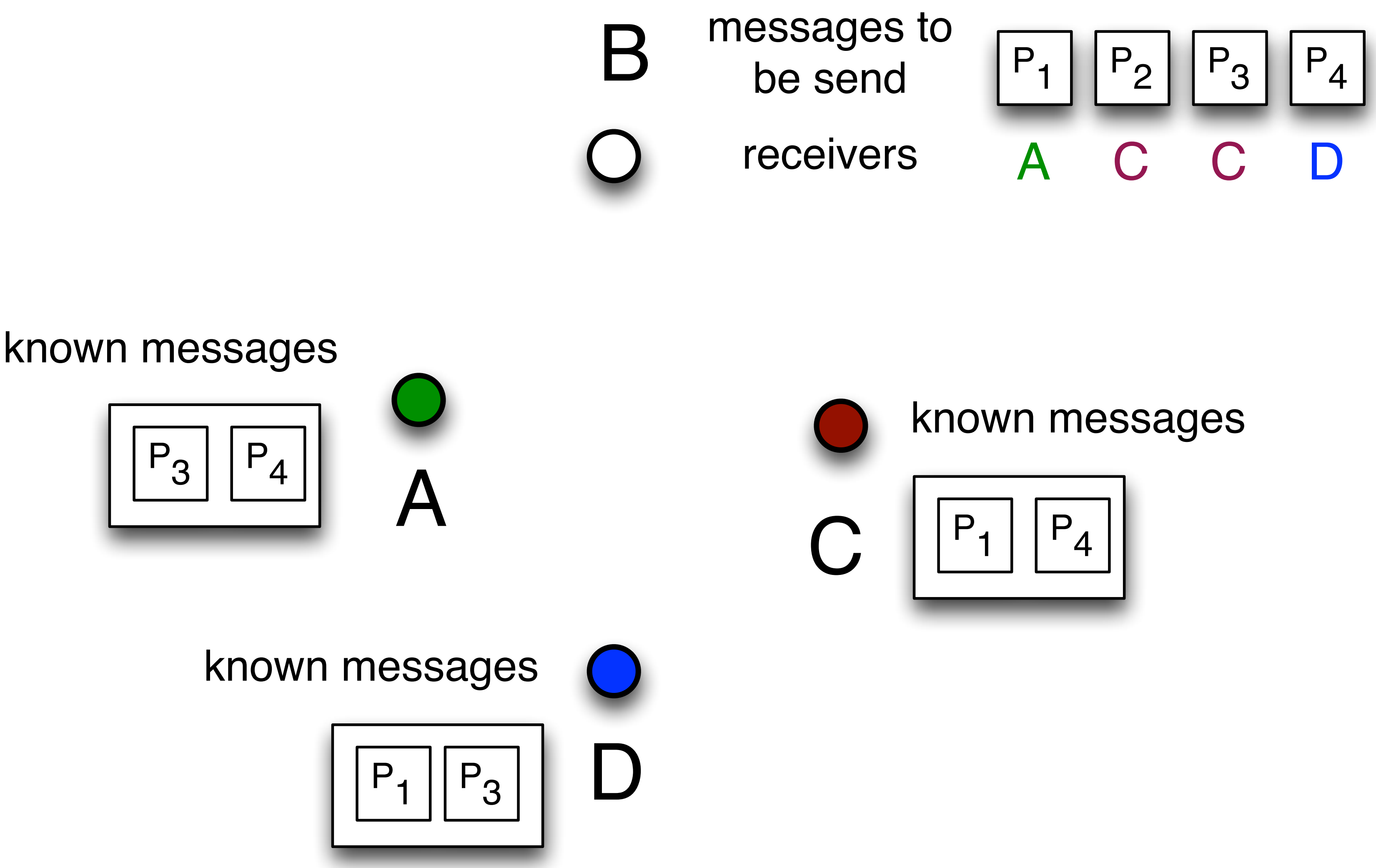
- **Problem**
  - Multihop messages cause interferences
- **Example**
  - Traditional: 4 messages to send
    - a message from A to B
    - and a message from B to A
  - Network Coding
    - 3 messages suffice



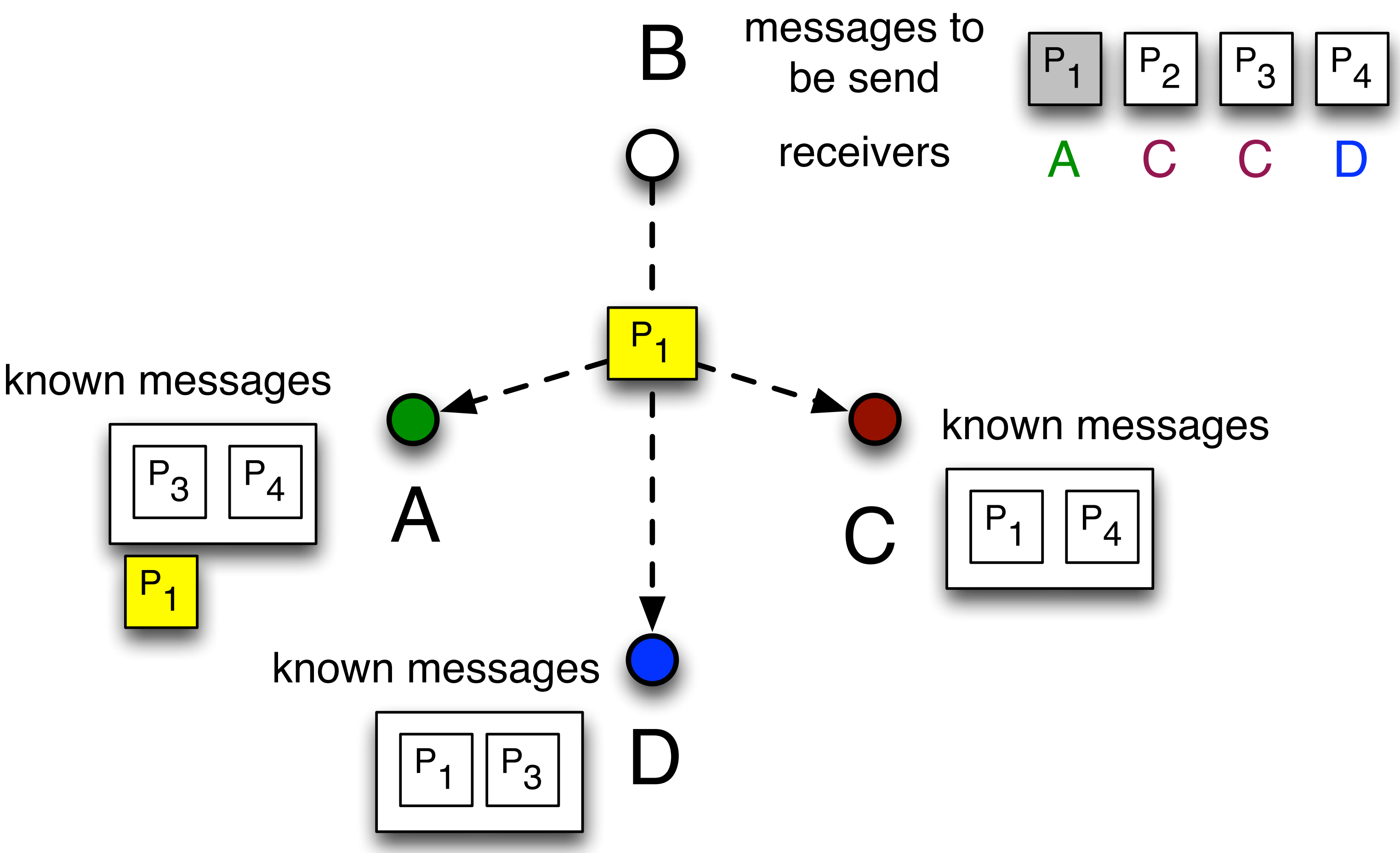
Coding Opportunistically COPE

- **Consider of multiple communication paths**
  - Opportunistic coding of messages by Xor
- **Utilization of the broadcast medium**
  - listening to the channel
  - all (even foreign) messages are buffered
  - buffered messages are used for decoding
- **Context messages**
  - announcement of level of knowledge
  - neighbors can generate code adapted to the receiver‘s knowledge
- **Guess the level of knowledge of neighbors**

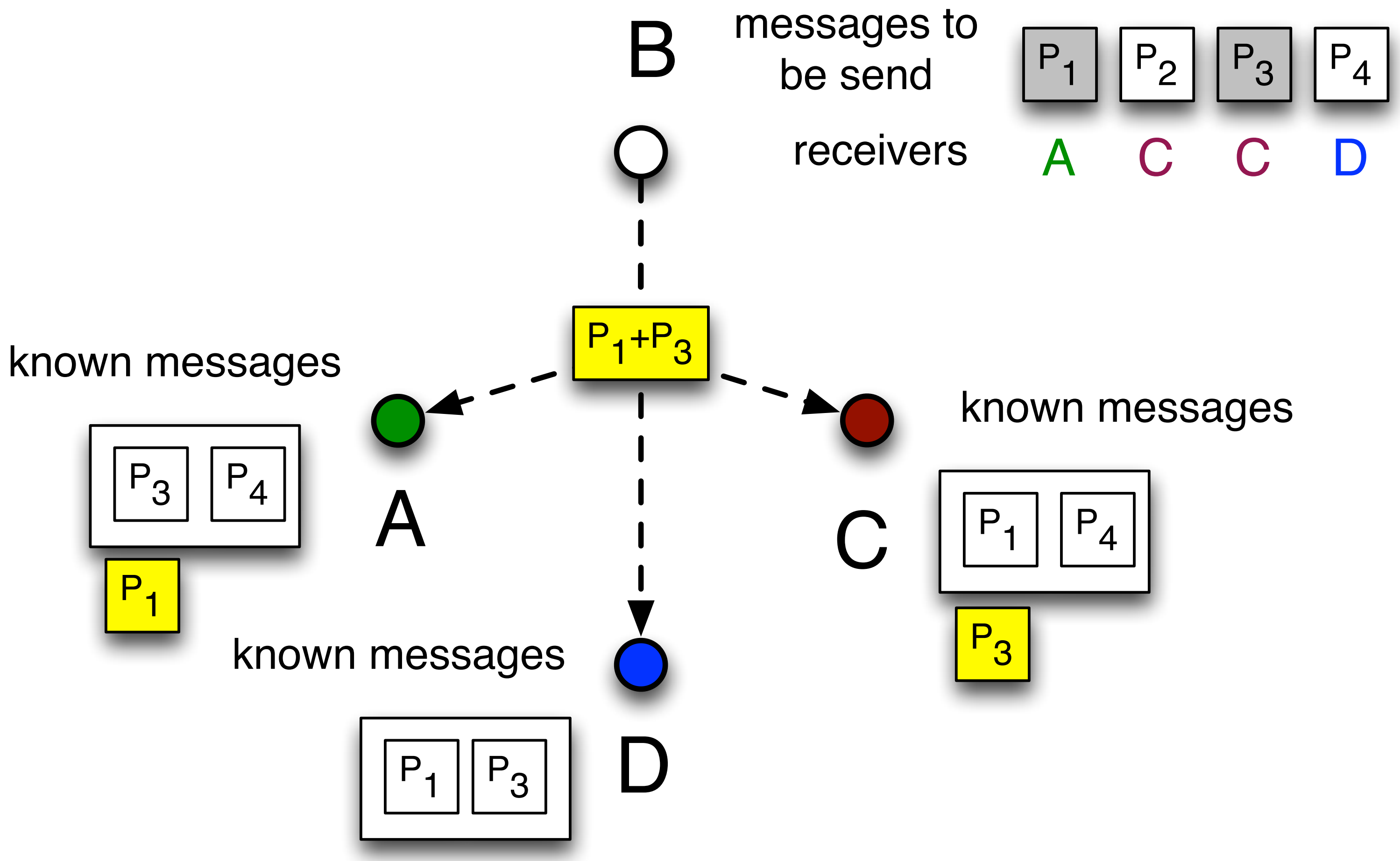
Opportunistic Coding



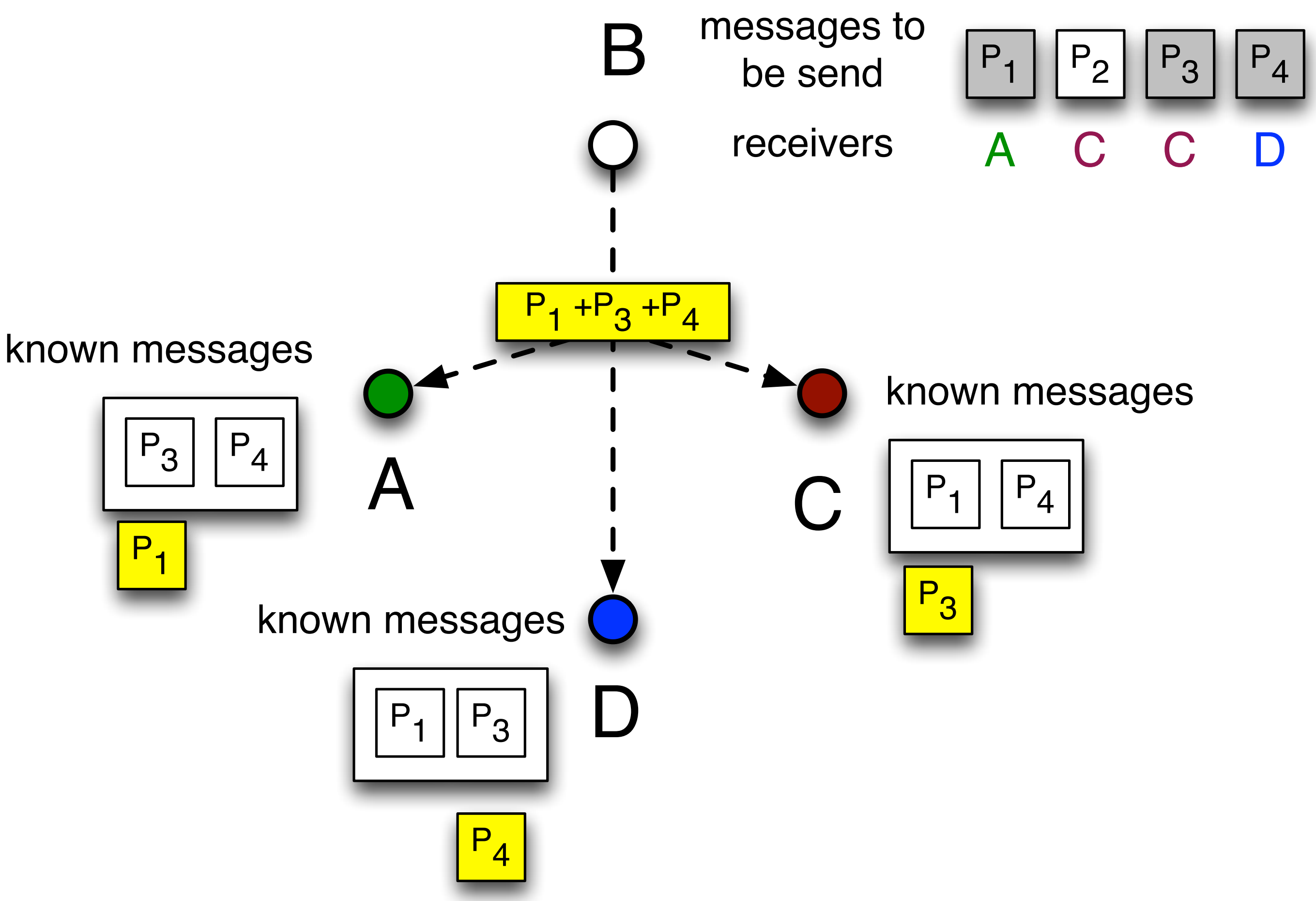
Opportunistic Coding

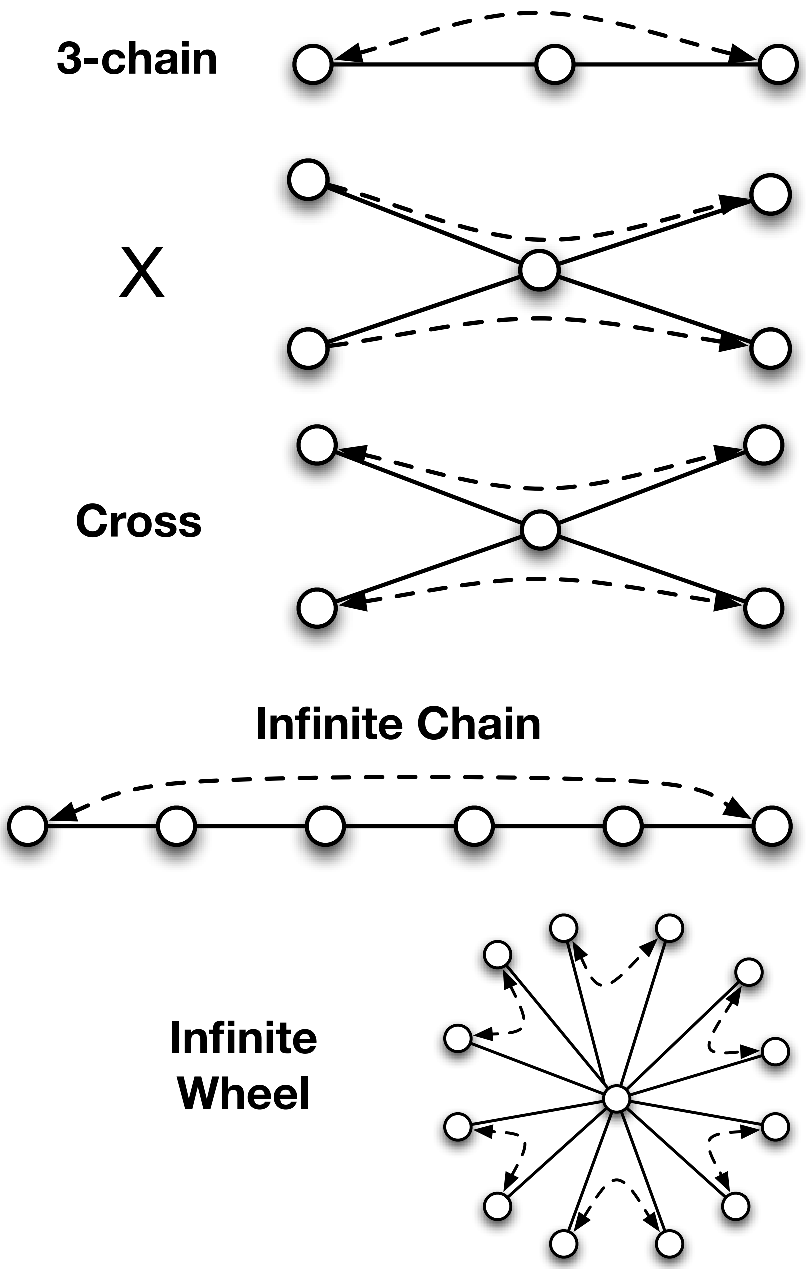


Opportunistic Coding



Opportunistic Coding





Coding Gain

Topology	Coding Gain
3-chain	1,333...
X	1,333...
Cross	1,666...
Infinite Chain	2
Infinite Wheel	2

Summary Network Coding

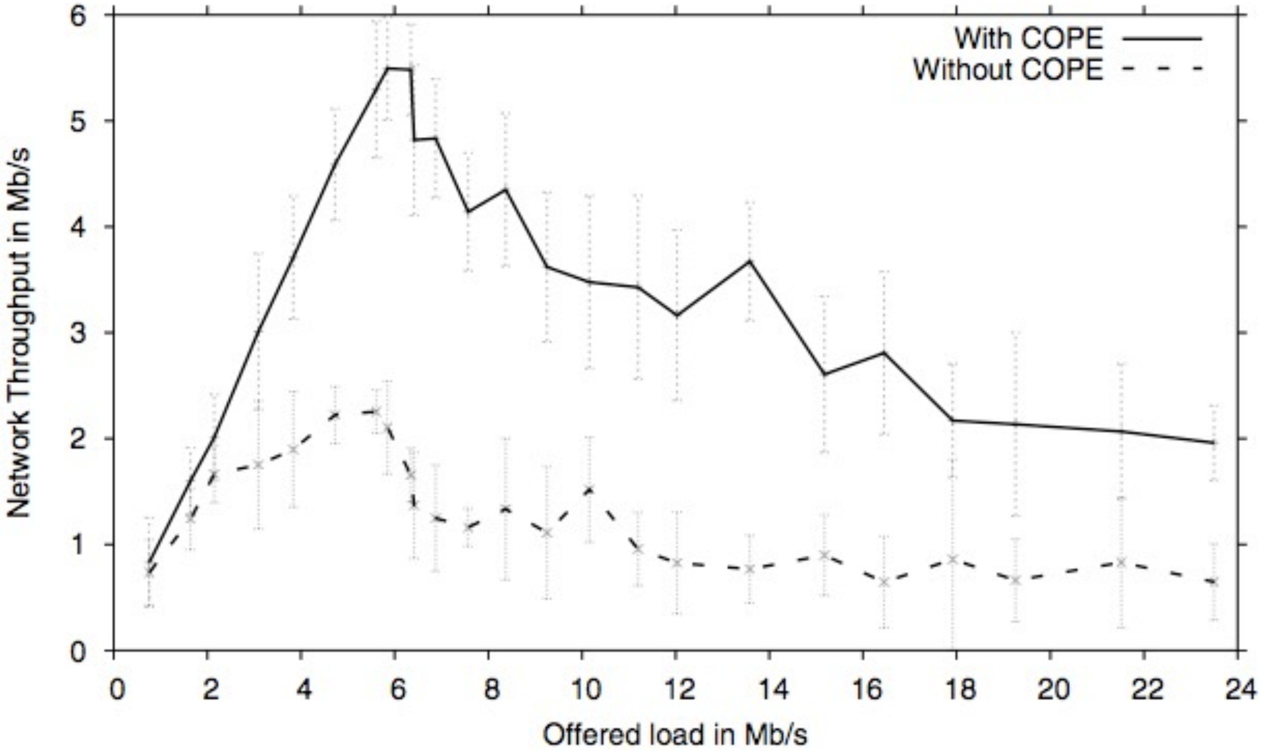


Figure 12—COPE can provide a several-fold (*3-4x*) increase in the throughput of wireless Ad hoc networks. Results are for UDP flows with randomly picked source-destination pairs, Poisson arrivals, and heavy-tail size distribution.

Wu, Chou, Sun-Yuan, Minimum-Energy Multicast in Mobile Ad hoc Networks using Network Coding, 2006

Network Coding

- **Benefit**
  - Network throughput can be increased
    - COPE
  - Reduction of energy consumption
  - Higher robustness, small error rate
  - Applications in peer-to-peer networks, wireless sensor networks
- **Problems**
  - complex encoding
  - sometimes high computational cost
  - difficult organization

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Algorithms for Radio Networks

Network Coding

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