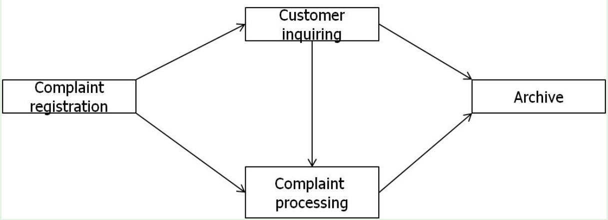
Chapter 12: Modeling and Analysis of Distributed Applications

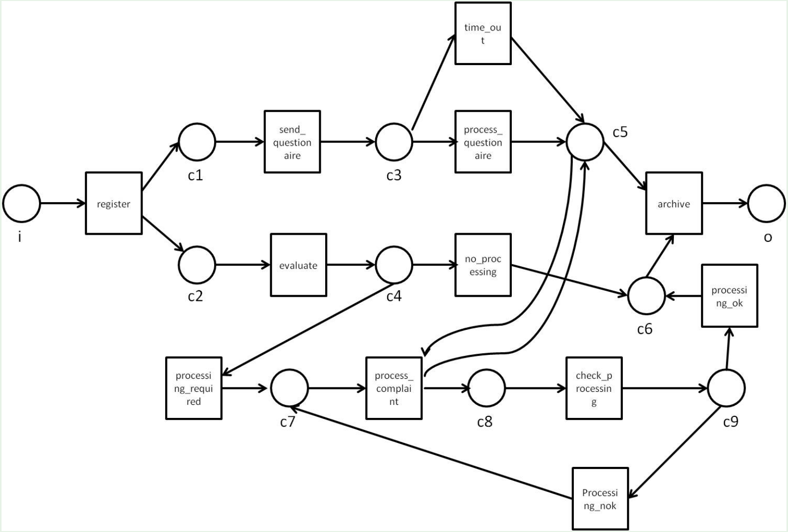
Petri-Nets

- Petri-nets are abstract formal models capturing the flow of information and objects in a way which makes it possible to describe distributed systems and processes at different levels of abstraction in a unified language.
- Petri-nets have the name from their inventor Carl Adam Petri, who introduced this formalism in his PhD-thesis 1962.

Processing of complaints: informal description.

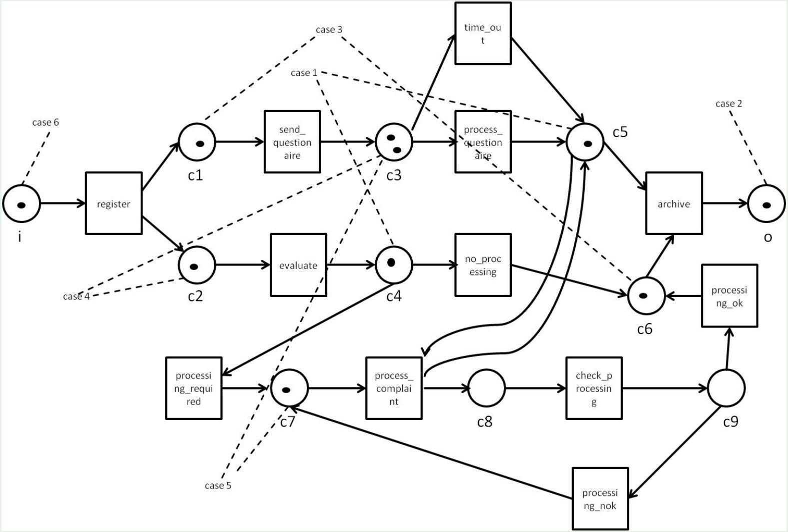


Complaints processing: formal Petri-net orchestration.[1]

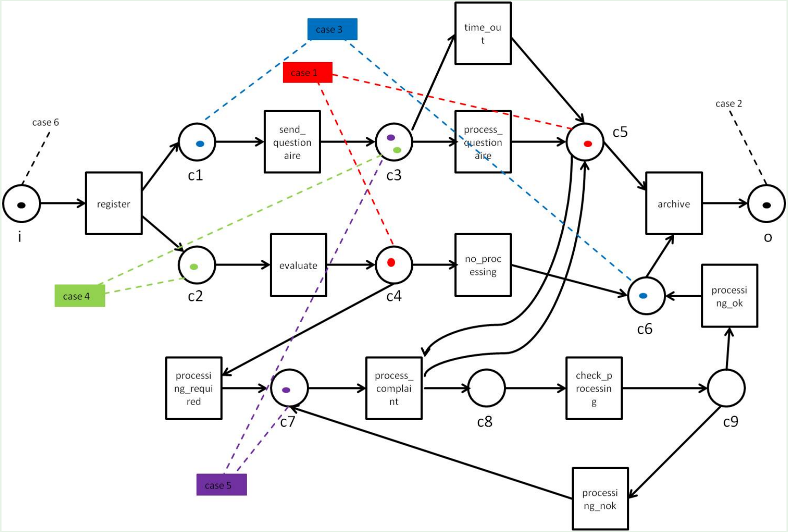


1 van der Aalst: The Application of Petri nets to Workflow Management. Journal of Circuits, Systems, and Computers 8(1): 21-66 (1998)

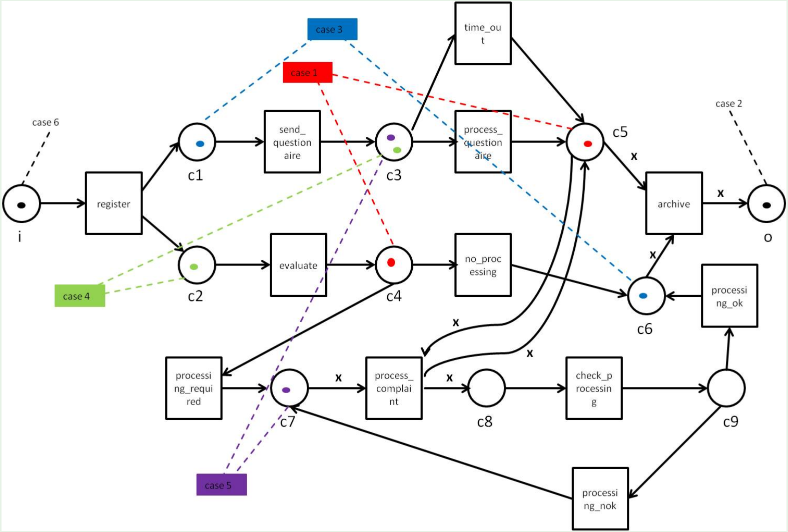
Complaints processing: more than one complaint



Complaints processing: how to distinguish complaints



Complaints processing: keeping things together



Petri-nets

Petri-nets model system dynamics.

- Activities trigger state transitions,
- activities impose control structures,
- applicable for modelling discrete systems.

Benefits

- Uniform language,
- can be used to model sequential, causual independent (concurrent, parallel, nondeterministic) and monitored exclusive activities.
- open for formal analysis, verification and simulation,
- graphical intuitive representation.

The name *Petri-net* denotes a variety of different versions of nets - we will discuss the special case of *System Nets* following the naming introduced by W. Reisig.

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Section 12.1 Elementary System Nets

Basic elements of an elementary System Net (eS-Net)

- System states are represented by *places*, graphically circles or ovals.
- A place may be marked by an arbitrary number of *tokens* graphically represented by black dots.
- System dynamics is represented by *transitions*, graphically rectangles.
- *Transitions* represent activities (events) and the causalities between such activities (events) are represented by edges.
- *Multiplicities* represent the consumption, respectively creation of resources which are caused by the *occurence* of activities.

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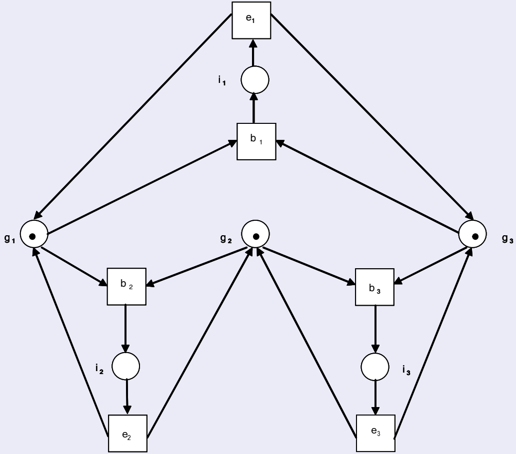
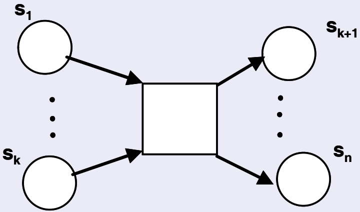
3-Philosopher-Problem

b_j: philosopher starts eating; e_j: philosopher stops eating;
i_j: philosopher is eating; g_j: fork on the desk;
$1 \leq j \leq 3$.



A transition *may* occur when certain conditions with respect to the markings of its directly connected places are fulfilled; the *occurence* of a transition - also called its *firing* - effects the markings of its directly connected edges, i.e. has local effects.

The *surrounding* of a transition t is given by t and all its directly connected places:



$s_1, \ldots, s_k$ are called *preconditions (pre-places)*, $s_{k+1}, \ldots, s_n$ *postconditions (post-places)*.

A place which is pre- and post-place at the same time is called a *loop*.

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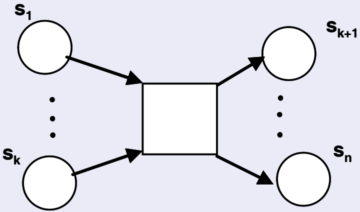

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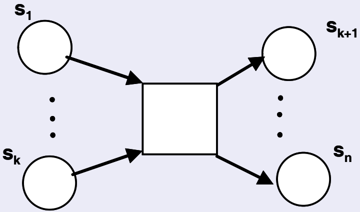

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A *net* is given as a tripel $N = (P, T, F)$, where

- P, the set of *places*, and T, the set of *transitionen*, are non-empty disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$, is the set of directed edges, called *flow relation*, which is a binary relation such that $dom(F) \cup cod(F) = P \cup T$.

Let $N = (P, T, F)$ be a net and $x \in P \cup T$.

$$xF := \{y \mid (x, y) \in F\}$$
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For $p \in P$, pF is the set of *post-transitions* of p; Fp is the set of *pre-transitions* of p.
For $t \in T$, tF is the set of *post-places* of t; Ft is the set of *pre-places* of t.

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Let $N = (P, T, F)$ be a net. Any mapping m from P into the set of natural numbers NAT is called a *marking* of P.

A mapping $P \to NAT \cup \{\omega\}$ is called ω-*marking*. ω represents an infinitly large number of tokens.

Arithmetic of ω:

$$\omega - n = \omega, \omega + n = \omega, n \cdot \omega = \omega, 0 \cdot \omega = 0, \omega > n$$

where $n \in NAT, n > 0$.

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A *eS-Net* is given as $N = (P, T, F, V, m_0)$, where

- (P, T, F) a net,
- $V : F \rightarrow NAT^+$ a *multiplicity*,
- m_0 a *marking* called *initial marking*.

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A transition may fire once it is enabled.

Let $N = (P, T, F, V, m_0)$ a eS-Net, m a marking and $t \in T$ a transition.

- *t is enabled at m*, if for all pre-places $p \in Ft$ there holds:

$$m(p) \geq V(p, t).$$

- Whenever t is enabled at m, then t may *fire* at m. Firing t at m transforms m to m', $m[\,t \succ m'$, in the following way:

$$m'(p) := \begin{cases} m(p) - V(p,t) + V(t,p) & \text{falls } p \in Ft, p \in tF, \\ m(p) - V(p,t) & \text{falls } p \in Ft, p \notin tF, \\ m(p) + V(t,p) & \text{falls } p \notin Ft, p \in tF, \\ m(p) & \text{sonst.} \end{cases}$$

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Transitions and markings in terms of vectors

Let places in P be linearily ordered.

- Markings of a net can be considered as vectors of nonnegative integers of dimension $\mid P \mid$, called *place-vectors*.
- Transitions t can be characterized as vectors of nonnegative integers of dimension $\mid P \mid$, called *transition vectors* $\Delta t, t^+, t^-$:

  Let $N = (P, T, F, V, m_0)$ a eS-Net, $p \in P$ and $t \in T$.

$$t^+(p) := \begin{cases} V(t,p) & \textit{if } p \in tF, \\ 0 & \textit{sonst.} \end{cases}$$

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Place and transition vectors at work:

- $m \leq m'$, if $m(p) \leq m'(p)$ for $\forall p \in P$,
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- t is enabled at m iff $t^- \leq m$,
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Reachability

Let $N = (S, T, F, V, m_0)$ a eS-Net.

We denote $W(T)$ the set of words with finite length over T; $\epsilon \in W(T)$ is called the *empty word*.

The length of a word $w \in W(T)$ is given by $l(w)$. We have $l(\epsilon) = 0$.

Let m, m' be markings of P and $w \in W(T)$. We define a relation $m[\, w \succ m'$ inductively:

- $m[\, \epsilon \succ m'$ iff $m = m'$,
- Let $t \in T, w \in W(T)$. $m[\, wt \succ m'$ iff $\exists m'' : m[\, w \succ m''$, $m''[\, t \succ m'$.

The *reachability relation* $[\, * \succ$of N is defined by

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- $R_N(m) := \{m' \mid m[*\succ m'\}$, the set of markings reachable from m by N,
- $L_N(m) := \{w \mid \exists m' : m[w\succ m'\}$, the set of all words representing firing sequences of transitions of N starting at m,
- $\Delta w := \sum_{i=1}^{n} \Delta t_i$, wobei $w = t_1 t_2 \ldots t_n$.

Results

- $[*\succ$ is reflexiv and transitiv.
- $m[w\succ m' \Rightarrow (m + m^*)[w\succ(m' + m^*), \forall m^* \in NAT^{|S|}$. (Monotonie)
- $m[w\succ m' \Rightarrow m' = m + \Delta w$.

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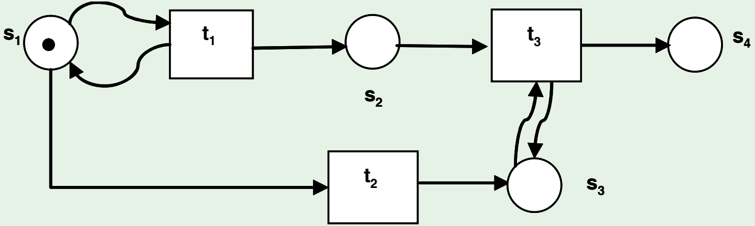
Reachability graph

Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Reachability graph* of N is a directed graph $EG(N) := (R_N(m_0), B_N)$; $R_N(m_0)$ is the set of nodes and B_N is the set of annotated edges as follows:

$$B_N = \{(m, t, m') \mid m, m' \in R_N(m_0), t \in T, m[\, t \succ m'\}.$$

Exercise: Give the reachability graph of the following eS-Net:



$$R_N(m_0) = \{ \quad (1,0,0,0),(1,1,0,0),(1,2,0,0),(1,3,0,0),\ldots,$$
$$(0,0,1,0),(0,1,1,0),(0,2,1,0),(0,3,1,0),\ldots,$$
$$(0,0,1,1),(0,1,1,1),(0,0,1,2),(0,2,1,1),(0,1,1,2),(0,0,1,3),\ldots\}$$

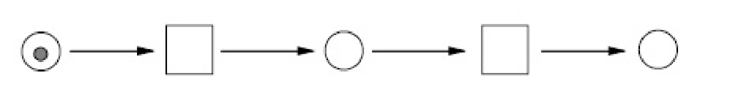
$$L_N(m_0) = \{ \quad \epsilon, t_1, t_1t_1, t_1t_1t_1, \ldots,$$
$$t_2, t_1t_2, t_1t_1t_2, t_1t_1t_1t_2, \ldots,$$
$$t_1t_2t_3, t_1t_1t_2t_3, t_1t_1t_2t_3t_3, t_1t_1t_1t_2t_3, t_1t_1t_1t_2t_3t_3, t_1t_1t_1t_2t_3t_3t_3, \ldots\}$$

Section 12.2 Control Patterns

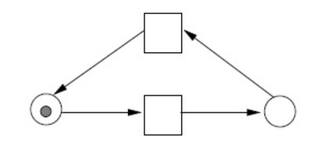
- eS-nets can be used to model *causal dependencies*; for modelling temporal aspects extensions of the formalism are required.
- Whenever between some transitions there are no causal dependencies, the transitions are called *concurrent*; concurrency is a prerequisite for parallelism.

Some typical causalities

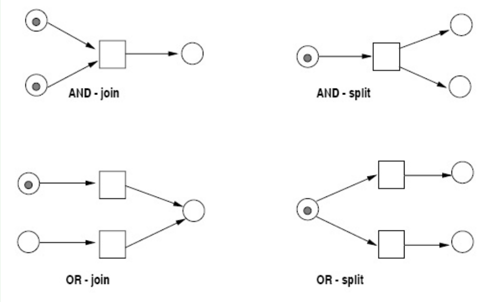
Sequence



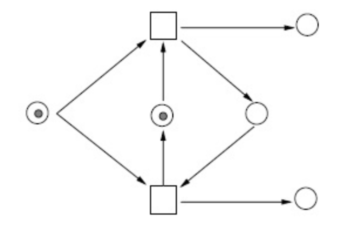
Iteration



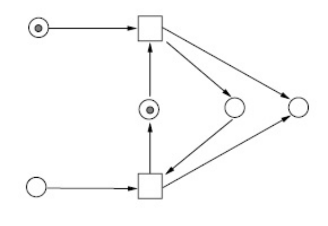
AND-join, OR-join, AND-split, OR-split



OR-Split with regulation



OR-Join with regulation



A eS-Net with concurrency