Section 7.3 Analysis

$m_0 = (1, 0)$

Boundedness

Let $N = (P, T, F, V, m_0)$ be a eS-Net, m a marking, $p \in P$.

- Let $k \in NAT^+$. p is called *k-bounded*, if for each marking m' there holds:

$$m' \in R_N(m_0) \Rightarrow m'(p) \leq k.$$

- p is called *bounded*, if p k-bounded for some $k \in NAT^+$.
- N is called *bounded (k-bounded)*, if each place is bounded (k-bounded).
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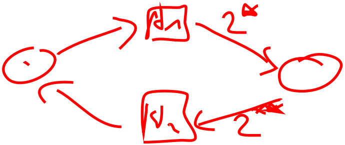

1-bounded
unbounded
2-bounded

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$w = t_1 t_2 t_3 t_1$ $\quad m[w \succ m'$

Theorem

Let $N = (P, T, F, V, m_0)$ be a eS-Net. N is *unbounded*, i.e. not bounded, iff there exist $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[\, w \succ m'$ and $m' > m$.

Proof $\Leftarrow$

Let $w \in W(T)$, $m, m' \in R_N(m_0)$, such that $m[\, w \succ m'$ and $m' > m$. It holds

$$m[\, w \succ m'[\, w \succ m''[\, w \succ m''' \ldots,$$

where $m < m' < m'' < m''' < \ldots$.

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Lemma

For each infinite sequence of markings (m_i) of markings there exists an infinite subsequence (m'_j), which is weakly monotonic, i.e. $l < k$ implies $m'_l \leq m'_k$.

To prove the Lemma, first extract an infinite subsequence for which weak monotonicity holds for the first components of its markings. Then extract from that subsequence an infinite subsequence for which weak monotonicity holds for the second components of its markings, etc.

$m_i,\ \underbrace{m_i',\ m_i'',\ \ldots}_{m_i \le m_2}$

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Proof $\Rightarrow$

- Consider the reachability graph $EG(N)$, which has an infinite number of nodes. Starting from m_0 there exist a directed path to each node of the graph. Because of the finite number of transitions, each node has only a finite number of direct successors.
- Thus, at m_0 there start an infinite number of paths without cycles, however only a finite number of edges. Therefore, one of these edges must be part of infinitly many paths. Let $m_0 \to m_1$ be one such edge.
- The same argument can be applied w.r.t. m_1 such that we get $m_0 \to m_1 \to m_2$, where $m_1 \to m_2$ is part of an infinite number of paths.
- The above construction can be repeated infinitly many times. Therefore there exists an infinite sequence of markings (m_i) of pairwise distinct markings, such that m_k, m_l, $0 \leq k \leq l$ implies:

  $$m_0[*\succ m_k[*\succ m_l.$$

  because of the Lemma there exists an infinite weakly monotonic subsequence (m'_j) von (m_i). Let m'_1, m'_2 two successive elements. From construction we have $m_0[*\succ m'_1[*\succ m'_2$, $m'_1 \leq m'_2$ and even $m'_1 < m'_2$.

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Let $N = (P, T, F, V, m_0)$ be a eS-Net, $m \in NAT^{|P|}$ a marking. The decision problem:

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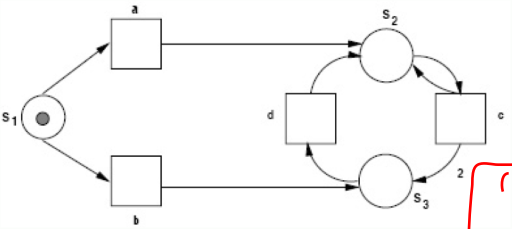
Consequence: Whenever a marking is not coverable w.r.t. some eS-Net N, it is not reachable in N.

Give examples.



$(1,0,0)$

a b

$(0,1,0) \xleftarrow{d} (0,0,1)$

c

$(0,1,2)$

c

$(0,1,4)$

$(0,1,\underline{\omega})$

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$m \leq m' \not\Rightarrow m$ is reachable

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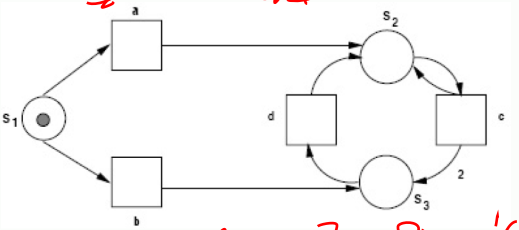

$\forall m' \in R_N(m_0): \exists p \in P: m'(p) < m(p)$

$\Rightarrow m$ is not reachable

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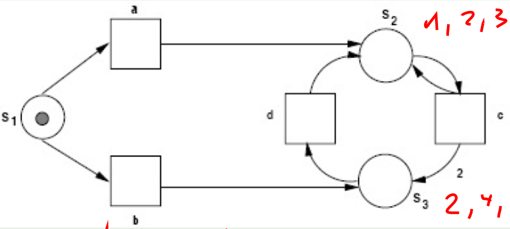

m = (0, 2, 3) (0, 2, 6)
m = (1, 2, 3)

Coverability Graph

Let $N = (P, T, F, V, m_0)$ a eS-Net. The *Coverability Graph* of N is given by $CG(N) := (R, B)$ as follows:

- *inductive definition of an auxiliary tree $T(N)$:*

  The values of the nodes in $T(N)$ are ω-markings of N. The value of the root node r is m_0. Let m be the value of some node n of $T(N)$, $t \in T$, and $m[t\succ m'$.

  - Whenever on the path from the root r to n there exists a node n'' with value m'' such that $m'' < m'$, then update m' by $m'(p) := \omega$ for all places p with $m''(p) < m'(p)$.
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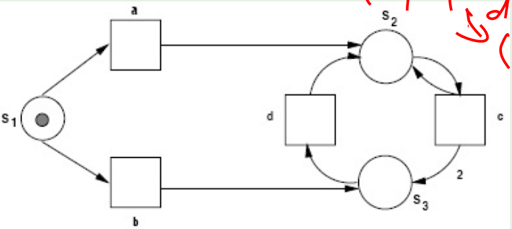
Give a coverability tree.

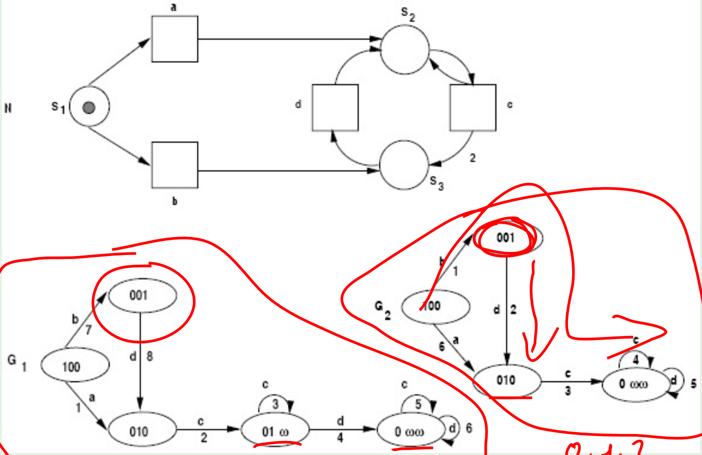


$(1,0,0) \xrightarrow{a} (0,1,0) \xrightarrow{c} (0,1,\omega) \xrightarrow{d} (0,\omega,\omega)$

$(1,0,0)$ a b

$(0,1,0) \xrightarrow{c} (0,1,?) \xrightarrow{c} (0,1,1) \rightarrow \dots$

A eS-net with two different coverability graphs.

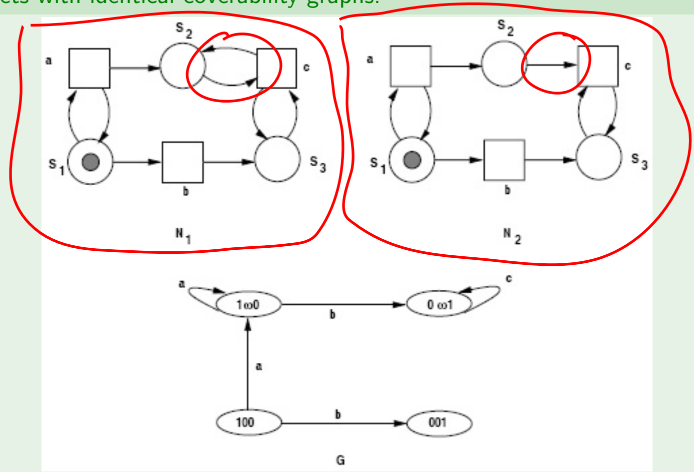


0, 1, 2 ω

(0, 1, 0) < (0, 1, 2)

(0, 0, 1) < (0, 1, 2)

Two eS-Nets with identical coverability graphs.



Theorem

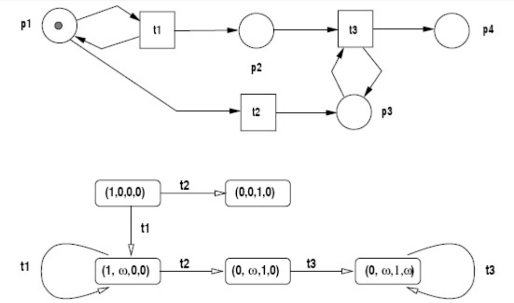
The coverability graph $CG(N) = (R, B)$ of a eS-net N is finite.

Proof:
Assume $CG(N)$ is not finite. Then it contains an infinite number of nodes. Thus there exists an infinite, weakly monotonic sequence of ω-markings, i.e. values of the nodes in the tree. Because of the construction of the auxiliary tree $T(N)$, such an infinite sequence cannot exist, as we can introduce ω only a finite number of times.

$$(\underset{\omega}{-}, \underset{\omega}{-}, \underset{\omega}{-})$$

To test the reachability of a certain marking we may first test its coverability and then try to find a firing sequence which confirms its reachability.

Is marking $m = (0, 3, 1, 3)$ reachable?



Yes, using the word $w = t_1^6 t_2 t_3^3$.

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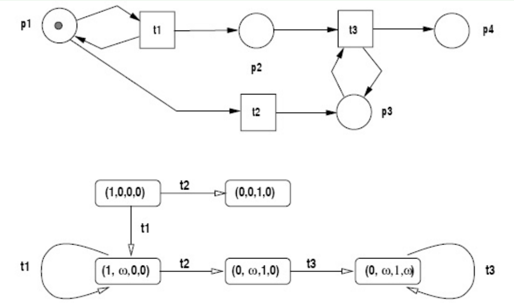

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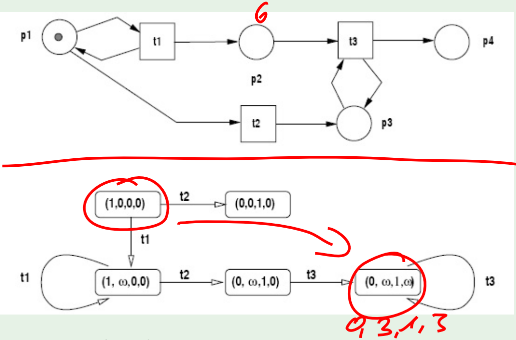

$m = (0,3,2,3)$ not reachable

Yes, using the word $w = t_1^6 t_2 t_3^3$.

$t_1^6 \; t_2 \; t_3^3$

Live, dead and deadlockfree

Let $N = (P, T, F, V, m_0)$ a eS-Net.

- A marking m is called *dead* in N, if there is no $t \in T$ which is enabled at m.
- A transition t is called *dead* at marking m, if there is no marking reachable from m, such that t is enabled.

  If t dead at m_0, then t is called dead in N.
- A transition t is called *live* at marking m, if for any reachable marking from m it holds that t is not dead.
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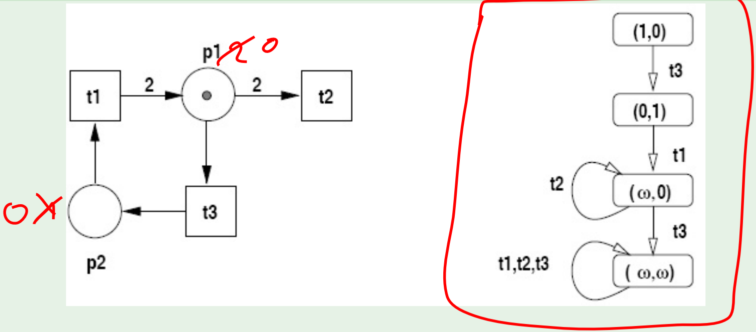

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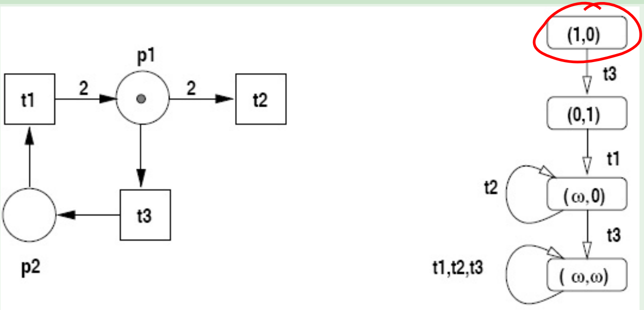

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Section 7.4 Invariants

Basics

- A Petri-net invariant is a property of a Petri-net, which holds for any marking, respectively transition word, of the net.
- We study place- and transition-invariants, which are based on a matrix representation of a net, respectively vector representation of markings and transitions.

Incidence Matrix

- Let $N = (P, T, F, V, m_0)$ a eS-Net, $T = \{t_1, \ldots, t_n\}$, $P = \{p_1, \ldots, p_m\}$, $n, m \geq 1$.
- A vector of dimension n (m) is called T- (P-)vector.
- For any $t \in T$, Δt can be represented as a column P-vector.
- The *incidence matrix* of N is given as a $m \times n$-matrix $C = (\Delta t_1, \ldots, \Delta t_n)$, respectively $C = (c_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, where $c_{ij} := \Delta t_j(s_i)$.

Section 7.4 Invariants

Basics

- A Petri-net invariant is a property of a Petri-net, which holds for any marking, respectively transition word, of the net.
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$$\begin{array}{c c} & \Delta t \\ \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} & \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \end{array}$$

Incidence Matrix

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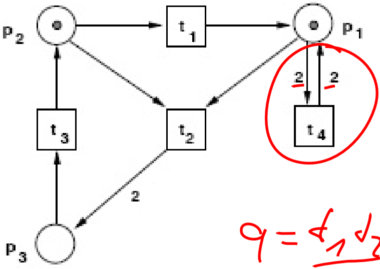
$\Delta t_1 = t_1^+ + t_1^-$

$m_0 = (1,1,0)$

Example



$$
\begin{array}{cc}
 & \begin{array}{cccc} \Delta t_1 & \Delta t_2 & \Delta t_3 & \Delta t_4 \end{array} \\
C = & \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array}
\end{array}
$$

$q = t_1 t_2 t_3 t_4$

$\bar{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$
C \cdot \bar{q} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array}
$$

- Incidence matrices are independent of concrete markings,
- In case of loops, information concerning multiplicities is lost.

Parikh-Vektor

The transpose of a vector x, resp. matrix C is denoted by $x^\top$, bzw. $C^\top$.

The *Parikh-Vektor* $\bar{q}$ of some $q \in W(T)$ is a column T-vector, $n = \mid T \mid$, defined as follows:

$$\bar{q} : T \rightarrow NAT, \text{ where } \bar{q}(t) \text{ is the number of occurences of } t \text{ in } q.$$

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$q = d_1 d_1 d_2 d_3 d_2$

$$\boxed{\bar{q} = \begin{array}{cl} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} & \begin{array}{l} d_1 \\ d_2 \\ d_3 \end{array} \end{array}}$$

State Equation

Let $q \in W(T)$ and m, m' markings.

If $m[\, q \succ m'$, then $\sum_{t \in T} (\bar{q}(t) \cdot \Delta t) = C \cdot \bar{q} = \Delta q$.

Moreover, as $m[\, q \succ m'$, we have

- $m' = m + \Delta q^{\top}$.

  The equation:

  $$m' = m + (C \cdot \bar{q})^{\top}$$

  is called *state equation*.

- The system of linear equations given by

  $$C \cdot x = (m' - m)^{\top}$$

  has an integer nonnegative solution x.

State Equation

$m' \in R_N(m)$

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however the following does not hold in general:

If $C \cdot x = (m' - m)^\top$ has an integer nonnegative solution then

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I.e., the reachability problem cannot be solved, in general.

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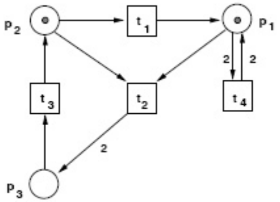

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

Let $m = (1, 0, 0)$, $m' = (0, 0, 1)$.
$x = (0, 1, 1, 0)^\top$ is a solution for $C \cdot x = (m' - m)^\top$, however we cannot find a word which can be fired at m.

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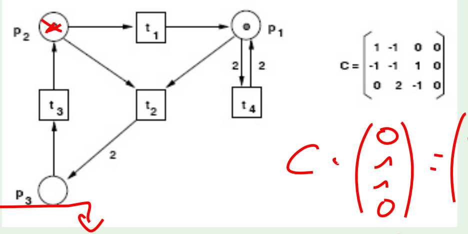

$C \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = (m' - m)^\top$

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Theorem

Let N be a eS-Net and Δ a P-vector. There exists a marking m^* and a word $q \in W(T)$, such that $m^*[\, q \succ (m^* + \Delta)$, iff $C \cdot x = \Delta^\top$ has an integer nonnegative solution.

Proof:
"$\Rightarrow$": trivial.

"$\Leftarrow$": Let $m^* := \sum_{t \in T} x(t) \cdot t^-$.

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$m(f(x)) > m' \qquad m' > \varepsilon m$

Corollary

Let $N = (P, T, F, V, m_0)$ be a eS-Net. There exists a marking m^* such that $N = (P, T, F, V, m^*)$ unbounded, iff $C \cdot x > 0$ has an integer nonnegative solution.

ω

Δx

Useful application of the corollary:

If there does not exist an integer nonnegative solution for $C \cdot x > 0$, then for any initial marking, N is bounded.

$\bar{x} > 0 \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} > 0$

$x_1, x_2, x_3 \geq 0$

$\exists x_i : x_i > 0$

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