Transition-Invariants (T-Invariants)

Let $N = (P, T, F, V, m_0)$ be a eS-Net.

- Any nontrivial integer solution $x$ of the homogenous linear equation system $C \cdot x = 0$ is called *transition-invariant* (*T-invariant*) of $N$.
- A T-invariant $x$ is called *proper*, if $x \geq 0$.
- A T-invariant $x$ is called *realizable* in $N$, if there exists a word $q \in W(T)$ with $\bar{q} = x$ and a reachable marking $m$ such that $m[q \succ m]$.
- $N$ is called *covered with T-invariants*, if there exists a T-invariant $x$ of $N$ with all components positive, i.e. greater than 0.

Proper T-invariants denote *possible* cycles of the reachability graph - realizable T-invariants denote cycles which indeed may occur.
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Proper T-invariants denote *possible* cycles of the reachability graph - realizable T-invariants denote cycles which indeed may occur.
Example

T-invariants of

\[
\begin{pmatrix}
1 \\
1 \\
2 \\
0
\end{pmatrix}
\]

\[+
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

are as follows:

\[x = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\]

where \(\lambda_1, \lambda_2\) integers.
Example

T-invariants of

\[ x = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

are as follows:

where \( \lambda_1, \lambda_2 \) integers.
Theorem

Let $N = (S, T, F, V, m_0)$ be a eS-Net. If there exists a marking $m$, such that $N$ live and bounded at $m$, then $N$ covered by T-invariants.

Proof: Let $N$ live and bounded at some $m$.

As $N$ is live at $m$, there exists a word $q_1 \in L_N(m)$, which contains all transitions in $T$ and the marking $m + \Delta q_1$ is reachable from $m$.

Moreover, $N$ is live at $m + \Delta q_1$ as well. Therefore, there exits a word $q_2 \in L_N(m)$, which contains all transitions in $T$ and $N$ is live at the marking $m + \Delta q_1 q_2$.

There exists an infinite sequence of markings $(m_i)$, where $m_i := m + \Delta q_1 \ldots q_i$, such that:

$$m[ q_1 \succ m_1[ q_2 \succ m_2 \ldots m_i[ q_{i+1} \succ m_{i+1} \ldots$$

As $N$ is bounded at $m$, there is only a finite number of markings which are reachable. Therefore, there exist $i, j \in \mathbb{N}^A : i < j$ such that $m_i = m_j$. Thus

$$m_i[ q_{i+1} \ldots q_j \succ m_j = m_i$$

As all these $q_i$ mention all transitions, we finally conclude

$$x = \bar{q}_{i+1} + \ldots + \bar{q}_j$$

is a T-Invariant which covers $N$. 
Theorem

Let $N = (S, T, F, V, m_0)$ be a eS-Net. If there exists a marking $m$, such that $N$ live and bounded at $m$, then $N$ covered by $T$-invariants.

Proof: Let $N$ live and bounded at some $m$.

As $N$ is live at $m$, there exists a word $q_1 \in L_N(m)$, which contains all transitions in $T$ and the marking $m + \Delta q_1$ is reachable from $m$.

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\[
m[q_1 > m_1[q_2 > m_2 \ldots m_i[q_{i+1} > m_{i+1} \ldots
\]

As \( N \) is bounded at \( m \), there is only a finite number of markings which are reachable. Therefore, there exist \( i, j \in \mathbb{N}^\ast : i < j \) such that \( m_i = m_j \). Thus

\[
m_i[q_{i+1} \ldots q_j > m_j = m_i
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Distributed Systems Part 2

Transactional Distributed Systems

Dr.-Ing. Thomas Hornung
Theorem

Let \( N = (S, T, F, V, m_0) \) be a eS-Net. If there exists a marking \( m \), such that \( N \) live and bounded at \( m \), then \( N \) covered by T-invariants.

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Moreover, \( N \) is live at \( m + \Delta q_1 \) as well. Therefore, there exits a word \( q_2 \in L_N(m) \), which contains all transitions in \( T \) and \( N \) is live at the marking \( m + \Delta q_1 q_2 \).

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\[
m[q_1 \succ m_1[q_2 \succ m_2 \ldots m_i[q_{i+1} \succ m_{i+1} \ldots]
\]

As \( N \) is bounded at \( m \), there is only a finite number of markings which are reachable. Therefore, there exist \( i, j \in \mathbb{N}^+ : i < j \) such that \( m_i = m_j \). Thus

\[
m_i[q_{i+1} \ldots q_j \succ m_j = m_i
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$m[q_1 \succ m_1[q_2 \succ m_2 \ldots m_i[q_{i+1} \succ m_{i+1} \ldots$ (infinite)

As $N$ is bounded at $m$, there is only a finite number of markings which are reachable. Therefore, there exist $i, j \in \mathbb{N}^+ : i < j$ such that $m_i = m_j$. Thus

$m_i[q_{i+1} \ldots q_j \succ m_j = m_i$ (finite)

As all these $q_i$ mention all transitions, we finally conclude

$x = \bar{q}_{i+1} + \ldots + \bar{q}_j$

is a $T$-Invariant which covers $N$. 

Distributed Systems Part 2
Transactional Distributed Systems
Dr.-Ing. Thomas Hornung
Useful application of the theorem:
Whenever $N$ is not covered by $T$-invariants, then for every marking it holds $N$ not live or not bounded.
Place-Invariants (P-Invariants)

Let $N = (P, T, F, V, m_0)$ be a eS-Net.

- Any nontrivial integer solution $y$ of the homogeneous linear equation system $y \cdot C = 0$ is called *place-invariant* (*P-invariant*) of $N$.
- A P-invariant $y$ is called *proper P-invariant*, if $y \geq 0$.
- $N$ is called *covered with P-invariants*, if there exists a P-invariant $y$ with all components positive, i.e. greater than 0.

If $y$ is a P-invariant, then for any marking $m$ the sum of the number of tokens on the places $p$ is invariant with respect to the firing of the transitions weighted by $y(p)$. 
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Example

P-invariants of

are as follows:

\[ y^T = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

where \( \lambda \) an integer.
Example

P-invariants of

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

are as follows:

\[
y^T = \lambda \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}^2
\]

where \(\lambda\) an integer.

\[
y^T \cdot \mathbf{m}^T = 2
\]

\[
y^T \cdot \mathbf{m'}^T = 2
\]
Theorem
Let $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{F}, \mathcal{V}, m_0)$ a eS-Net and let $y$ a P-invariant of $\mathcal{N}$. Then:

$$m \in R_{\mathcal{N}}(m_0) \Rightarrow y \cdot m^\top = y \cdot m_0^\top.$$

Proof:
Assume $m_0 \triangleright m$. Then $m = m_0 + (C \cdot \bar{q})^\top$ and also:

$$y \cdot m^\top = y \cdot m_0^\top + y \cdot (C \cdot \bar{q}) =$$

$$= y \cdot m_0^\top + (y \cdot C) \cdot \bar{q} = y \cdot m_0^\top + 0 \cdot \bar{q} = y \cdot m_0^\top.$$
Theorem

Let \( N = (P, T, F, V, m_0) \) a eS-Net and let \( y \) a P-invariant of \( N \). Then:

\[
m \in R_N(m_0) \Rightarrow y \cdot m^\top = y \cdot m_0^\top.
\]

**Proof:**

Assume \( m_0[q > m] \). Then \( m = m_0 + (C \cdot \bar{q})^\top \) and also:

\[
y \cdot m^\top = y \cdot m_0^\top + y \cdot (C \cdot \bar{q}) =
\]

\[
= y \cdot m_0^\top + (y \cdot C) \cdot \bar{q} = y \cdot m_0^\top + 0 \cdot \bar{q} = y \cdot m_0^\top.
\]
Corollary:

- Let $y$ P-invariante of $N$, $m$ marking.
  
  $$y \cdot m^T \neq y \cdot m_0^T \Rightarrow m \not\in R_N(m_0).$$

- Let $y$ proper P-invariant of $N$. Let $p \in P$ such that $y(p) > 0$.

  Then, for any initial marking, $p$ is bounded.

  Proof: $y \cdot m_0^T = y \cdot m^T \geq y(p) \cdot m(p) \geq m(p)$.

- Let $N$ be covered by P-invariants. $N$ is bounded for any initial marking.
Corollary:

- Let $y$ P-invariant of $N$, $m$ marking.
  
  $y \cdot m^T \neq y \cdot m_0^T \Rightarrow m \not\in R_N(m_0)$.

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Corollary:

- Let $y$ a P-invariant of $N$, $m$ marking.
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- Let $y$ proper P-invariant of $N$. Let $p \in P$ such that $y(p) > 0$.
  Then, for any initial marking, $p$ is bounded.

  Proof: $y \cdot m_0^T = y \cdot m^T \geq y(p) \cdot m(p) \geq m(p)$.

- Let $N$ be covered by P-invariants. $N$ is bounded for any initial marking.
Note, the following net is bounded for any initial marking, however does not have a P-invariant:

![Petri-Net Example](image)

P-invariants allow sufficient tests for non-reachability and boundedness.
Note, the following net is bounded for any initial marking, however does not have a P-invariant:

\[ 100 \div 93 \]

P-invariants allow sufficient tests for non-reachability and boundedness.
Example: Prove freedom from deadlocks.

Initial marking is given by \( m_0 = (2, 0, 0, 0, 1, 1, 1) \). Assume there exist a dead marking \( m, m_0 \models q \succ m \). Then it must hold \( m(p_1) = m(p_2) = m(p_3) = 0 \). Because of \( Y_4 \) it follows \( m(p_0) = 2 \). As \( m \) dead it follows \( m(p_4) = m(p_5) = m(p_6) = 0 \). However this contradicts \( Y_1m_0 = Y_1m \).
Example: Prove freedom from deadlocks.

\[ C = \begin{bmatrix}
-1 & -1 & -1 & -1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

P-invariants:

\[ Y_1 = (0, 1, 0, 0, 1, 0, 0) \]
\[ Y_2 = (0, 0, 1, 0, 0, 1, 0) \]
\[ Y_3 = (0, 0, 0, 1, 0, 0, 1) \]
\[ Y_4 = (1, 1, 1, 1, 0, 0, 0) \]

Initial marking is given by \( m_0 = (2, 0, 0, 0, 1, 1, 1) \). Assume there exist a dead marking \( m, \ m_0 \not\succ m \). Then it must hold \( m(p_1) = m(p_2) = m(p_3) = 0 \). Because of \( Y_4 \) it follows \( m(p_0) = 2 \). As \( m \) dead it follows \( m(p_4) = m(p_5) = m(p_6) = 0 \). However this contradicts \( Y_1 m_0 = Y_1 m \).
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0 & 0 & 0 & 0 & 1 & 0
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\[ m = (2, 0, 0, 0, 0, 0, 0) \]

\( m \) is not reachable.
Section 7.5 Place Capacities

Sometimes when modelling we would like to fix an upper bound for the number of tokens in a place.

- Let \( N = (P, T, F, V, m_0) \) be a eS-Net, \( c \) a \( \omega \)-marking of \( P \) and let \( m_0 \leq c \). 
  \((N, c)\) is called eS-Net with capacities. \( c(p), p \in P \) is called capacity of \( p \).

- For eS-nets with capacities the notion of being enabled is adapted:
  
  \[ a \text{ transition } t \in T \text{ is enabled at marking } m, \text{ if } t^- \leq m \text{ and } m + \Delta t \leq c. \]

- Capacities graphically are labels of places - no label means capacity \( \omega \).
Section 7.5 Place Capacities

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- Capacities graphically are labels of places - no label means capacity $\omega$. 
Any eS-net with capacities can be simulated by a eS-Net without capacities.

**Construction**

- Let $p$ a place with capacity $k = c(p), k \geq 1$. Let $p^{co}$ be the complementary place of $p$ which is assigned the initial marking $k - m_0(p)$.

- Whenever for a transition $t$ we have $\Delta t(p) > 0$, we introduce an arc from $p^{co}$ to $t$ with multiplicity $\Delta t(p)$; whenever $\Delta t(p) < 0$, we introduce an arc from $t$ to $p^{co}$ with multiplicity $-\Delta t(p)$. 
Any eS-net with capacities can be simulated by a eS-Net without capacities.

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Construction

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  whenever $\Delta t(p) < 0$, we introduce an arc from $t$ to $p^{co}$ with multiplicity $-\Delta t(p)$. 
A eS-Net with capacities and its simulation by a bounded eS-Net.

\[ N_1(k = 4): \]

\[ N_2: \]

\[ 4 \leq 1 = 3 \]
Section 7.6 S-Nets with Colors

- eS-Nets in practice may become huge and difficult to understand.
- Sometimes such nets exhibit certain regularities which give rise to questions how to reduce the size of the net without losing modeling properties.
What about a $n$-philosopher problem with $n >> 3$?

Why not introduce tokens with individual information?
What about a \( n \)-philosopher problem with \( n \gg 3 \)?

Why not introduce tokens with individual information?
Abstraction 5-philosopher problem

Note: the intention of the marking shown only is to demonstrate „individual“ tokens.

What about being enabled and firing?
Abstraction 5-philosopher problem

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What about being enabled and firing?
Colored System-Nets

A colored System-Net distinguishes different kinds of sorts for markings - the so called *colors* - and functions over these sorts which are used to label the edges of the net.

Generalizing eS-Nets, in a colored net a transition will be called enabled, if certain conditions are true, which are based on the functions which are assigned to the edges of the transitions surrounding.

Thus, we have colors, to characterize markings (*place colors*), and colors, to characterize the firing of transitions (*transition colors*).

As a marking of a place now can be built out of different kind of tokens, we introduce multisets.

- Let $A$ be a set. A *multiset* $m$ over $A$ is given by a mapping $m : A \rightarrow \text{NAT}$.
- Let $a \in A$. If $m[a] = k$ then there exist $k$ occurrences of $a$ in $m$.
- A multiset oftenly is written as a (formal) sum, e.g. $[\text{Apple}, \text{Apple}, \text{Pear}]$ is written as $2 \cdot \text{Apple} + 1 \cdot \text{Pear}$. 
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A colored version of the 3-Philosopher-Problem

Colors

\[ C(g) = \{g_1, g_2, g_3\}, \quad C(i) = \{ph_1, ph_2, ph_3\} \quad \text{place colors} \]

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Functions

\[ ID(ph_j) := 1 \cdot ph_j, 1 \leq j \leq 3 \]

\[ RL(ph_j) := \begin{cases} 
1 \cdot g_1 + 1 \cdot g_3 & \text{if } j = 1, \\
1 \cdot g_{j-1} + 1 \cdot g_j & \text{if } j \in \{2, 3\}.
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**Multiplicities**

A *multiplicity* assigned to an edge between a place $p$ and a transition $t$ is a mapping from the set of transition colors of $t$ into the set of multisets over the colors of $p$.

In the example:

$$V(b, i) = V(i, e) = ID, \ V(g, b) = V(e, g) = RL,$$

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$ID$ denotes the identity mapping.

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Markings are multisets over the respective place colors.

In the example:

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A colored Net $CN = (P, T, F, C, V, m_0)$ is given by:

- A net $(P, T, F)$.
- A mapping $C$ which assigns to each $x \in P \cup T$ a finite nonempty set $C(x)$ of colors.
- Mapping $V$ assignes to each edge $f \in F$ a mapping $V(f)$.

Let $f$ be an edge connecting place $p$ and transition $t$. $V(f)$ is a mapping from $C(t)$ into the set of multisets over $C(p)$.

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- A transition \( t \) is enabled in color \( d \in C(t) \) at \( m \), if for all pre-places \( p \in Ft \) there holds:

\[
V(p, t)(d) \leq m(p).
\]

- Assume \( t \) is enabled in color \( d \) at marking \( m \). Firing of \( t \) in color \( d \) transforms \( m \) to a marking \( m' \):

\[
m'(p) := \begin{cases} 
m(p) - V(p, t)(d) + V(t, p)(d) & \text{if } p \in Ft, p \in tF, \\
m(p) - V(p, t)(d) & \text{if } p \in Ft, p \notin tF, \\
m(p) + V(t, p)(d) & \text{if } p \notin Ft, p \in tF, \\
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Fold and Unfold of a Colored System-Net

Folding

By folding of a eS-Net we can reduce the number of places and transitions; places and transitions are represented by appropriate place and transition colors, on which certain functions defining the multiplicities are defined.

Let \( N = (P, T, F, V, m_0) \) a eS-Net. A folding is defined by \( \pi \) and \( \tau \):

- \( \pi = \{q_1, \ldots, q_k\} \) a (disjoint) partition of \( P \),
- \( \tau = \{u_1, \ldots, u_n\} \) a (disjoint) partition of \( T \).
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$\mathbb{Q}$

- $q_1 = \{p_1, p_3, p_4\}$
- $q_2 = \{p_2, p_1, r_8\}$
- $q_3 = \ldots$

$\bigcup_{i=1}^{\mathbb{Q}} q_i = P$
Two special cases

Call $GN(\pi, \tau) := (P', T', F', C', V', m'_0)$ the result of folding.

- All elements of $\pi, \tau$ are one-elementary:
  \[ \Rightarrow N \text{ and } GN(\pi, \tau) \text{ are isomorphic}, \]

- $\pi, \tau$ contain only one element:
  \[ \Rightarrow |P'| = |T'| = 1, \text{”the model is represented by the labellings”}. \]
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Two special cases

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Folding $\pi = \{\{g_1, g_2, g_3\}, \{i_1, i_2, i_3\}\}$, $\tau = \{\{b_1, b_2, b_3\}, \{e_1, e_2, e_3\}\}$.

Colors from folding:
$C(g) = \{g_1, g_2, g_3\}$, $C(i) = \{i_1, i_2, i_3\}$, $C(b) = \{b_1, b_2, b_3\}$, $C(e) = \{e_1, e_2, e_3\}$

Multiplicities: $ID, RL$ analogously to previous version.
3-Philosopher-Problem

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Multiplicities: \( ID, RL \) analogously to previous version.
3-Philosopher-Problem?

\[ \pi = \{P\}, \tau = \{T\}, \]
\[ S' = \{s'\}, T' = \{t'\}, \]
\[ C(s') = \{g_1, g_2, g_3, i_1, i_2, i_3\}, \]
\[ C(t') = \{b_1, b_2, b_3, e_1, e_2, e_3\}, \]
\[ m_0'(s') = g_1 + g_2 + g_3, \]

\[ V'(s', t')(t) = \begin{cases} 
  g_1 + g_3 & \text{falls } t = b_1, \\
  g_1 + g_2 & \text{falls } t = b_2, \\
  g_2 + g_3 & \text{falls } t = b_3, \\
  i_1 & \text{falls } t = e_1, \\
  i_2 & \text{falls } t = e_2, \\
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\end{cases} \]

\[ V'(t', s')(t) = \begin{cases} 
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  g_2 + g_3 & \text{falls } t = e_3, \\
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Given $\pi = \{q_1, \ldots, q_k\}$, $\tau = \{u_1, \ldots, u_n\}$.

The folding $GN(\pi, \tau) := (P', T', F', C', V', m'_0)$ of $N$ is defined as follows:

- $P' := \{p'_1, \ldots, p'_k\}$; $T' := \{t'_1, \ldots, t'_n\}$.
- $C'(p'_i) = q_i$ für $i = 1, \ldots, k$; $C'(t'_j) = u_j$ für $j = 1, \ldots, n$.
- $F' := \{(p', t') | C'(p') \times C'(t') \cap F \neq \emptyset\} \cup \{(t', p') | C'(t') \times C'(p') \cap F \neq \emptyset\}$.
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- $F' := \{(p', t') | C'(p') \times C'(t') \cap F \neq \emptyset\} \cup \{(t', p') | C'(t') \times C'(p') \cap F \neq \emptyset\}$,
- $f' = (p', t') \in F'$: $V'(f')$ is defined ($t \in C'(t')$):
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- $m'_0(p') := \sum_{p \in C'(p')} m_0(p) \cdot p$. 
Given \( \pi = \{q_1, \ldots, q_k\}, \tau = \{u_1, \ldots, u_n\} \).

The folding \( GN(\pi, \tau) := (P', T', F', C', V', m'_0) \) of \( N \) is defined as follows:

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\[ C(s_1') = \{ s_1, s_2 \} \]
\[ C(s_2') = \{ s_3 \} \]
\[ g(t_1) = \sqrt{s_1 + 2s_2} \]
\[ g(t_2) = s_2 \]
\[ m_0'(s_1') = s_1 + 2s_2 \]
\[ m_0'(s_2') = 0 \]
\[ g(t_1) = 2s_3 \]
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Unfolding

Let $GN = (P, T, F, C, V, m_0)$ a CN-Net.

The Unfolding of $GN$ is an eS-Net $GN^* := (P^*, T^*, F^*, V^*, m_0^*)$ given as follows:

- $P^* := \{(p, c) \mid p \in P, c \in C(p)\}$,
- $T^* := \{(t, d) \mid t \in T, d \in C(t)\}$,
- $F^* := \{((p, c), (t, d)) \mid (p, t) \in F, V(p, t)(d)[c] > 0\} \cup \{((t, d), (p, c)) \mid (t, p) \in F, V(t, p)(d)[p] > 0\}$.
- $V^*((p, c), (t, d)) := V(p, t)(d)[c]$,
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\[ m_0'(s'_1) = s_1 + 2s_3 \]
\[ m_0'(s_2') = 0 \]
\[ (s_1', s_3') \rightarrow (d_1', \phi) \]
\[ V(s_1', d_1') (d_2') s_3 = 0 \]
**Definition**

Let $E$ be a certain property of a net, e.g. boundedness, liveness, or reachability.

A CS-Net $GN$ has property $E$, whenever its unfolding $GN^*$ has property $E$.

**Analysis of colored System Nets**

- **Analyse unfolding:**
  - **Advantage:** Methods exist,
  - **Pitfall:** Unfoldings may be huge eS-Nets.

- **Analyse colored net:**
  - Reachability graph and coverability graph can be defined in analogous way to eS-Nets.
  - There exists a theory for invariants, as well.
  - Tools for simulation and analysis are available.
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