ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG

Algorithms for Radio Networks

Localization

University of FreiburgTechnical Faculty
Computer Networks and Telematics
Prof. Christian Schindelhauer

CoNe Freiburg

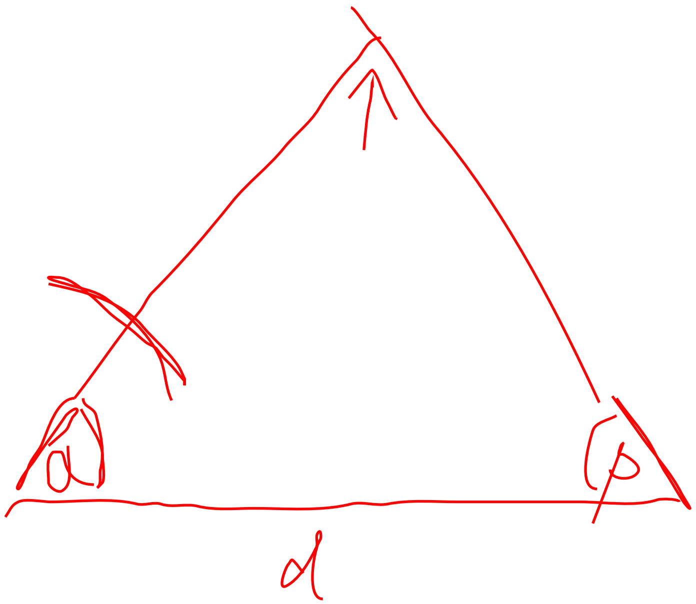
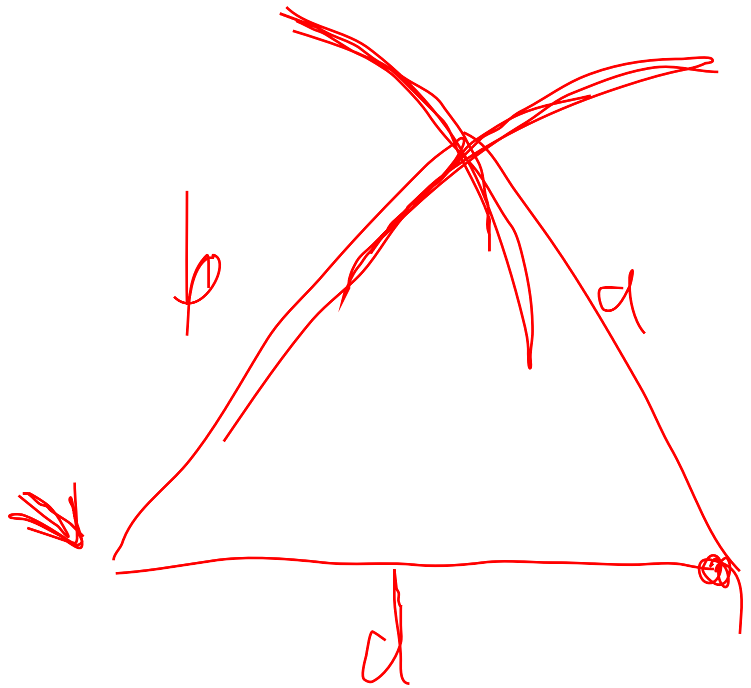
IIF
INSTITUT FÜR INFORMATIK FREIBURG

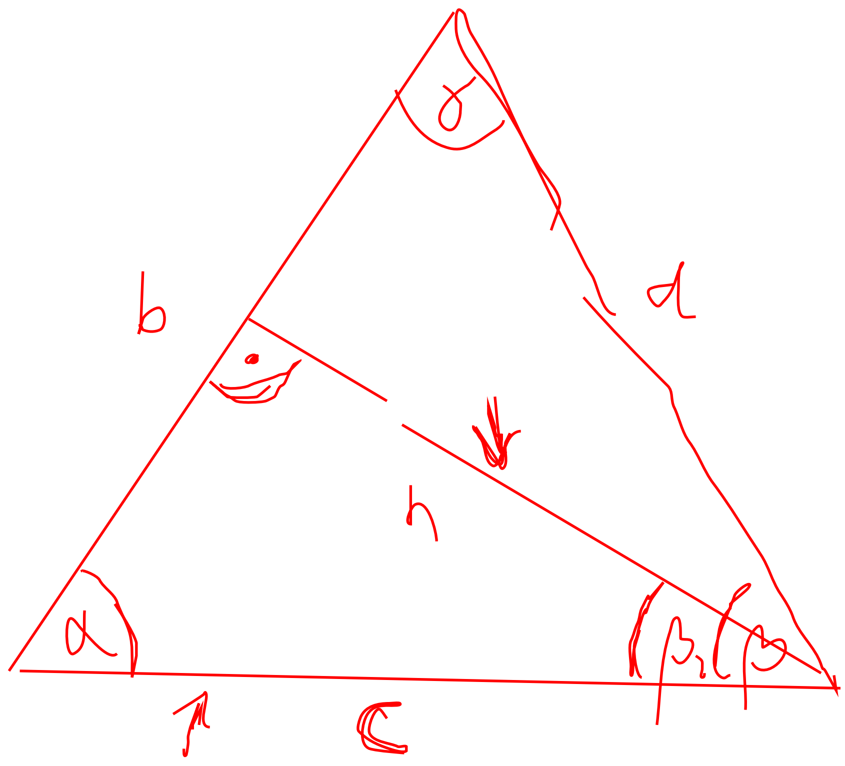
Trilateration

- **Assuming the distance to three points is given**
- **System of equations**
  - **(x_i, y_i): coordinates of an anchor point i,**
  - **r distance from the anchor point i**
  - **(x_u, y_u): unknown coordinates of a node**

$$(x_i - x_u)^2 + (y_i - y_u)^2 = r_i^2 \text{ for } i = 1, \ldots, 3$$

- **Problem: Quadratic equations**
  - **Transformations lead to a linear system of equations**


d
β

β
d



$$\frac{h}{c} = \cos \beta_1$$

$$\frac{h}{a} = \cos \beta_2$$

Trilateration

- **System of equations**

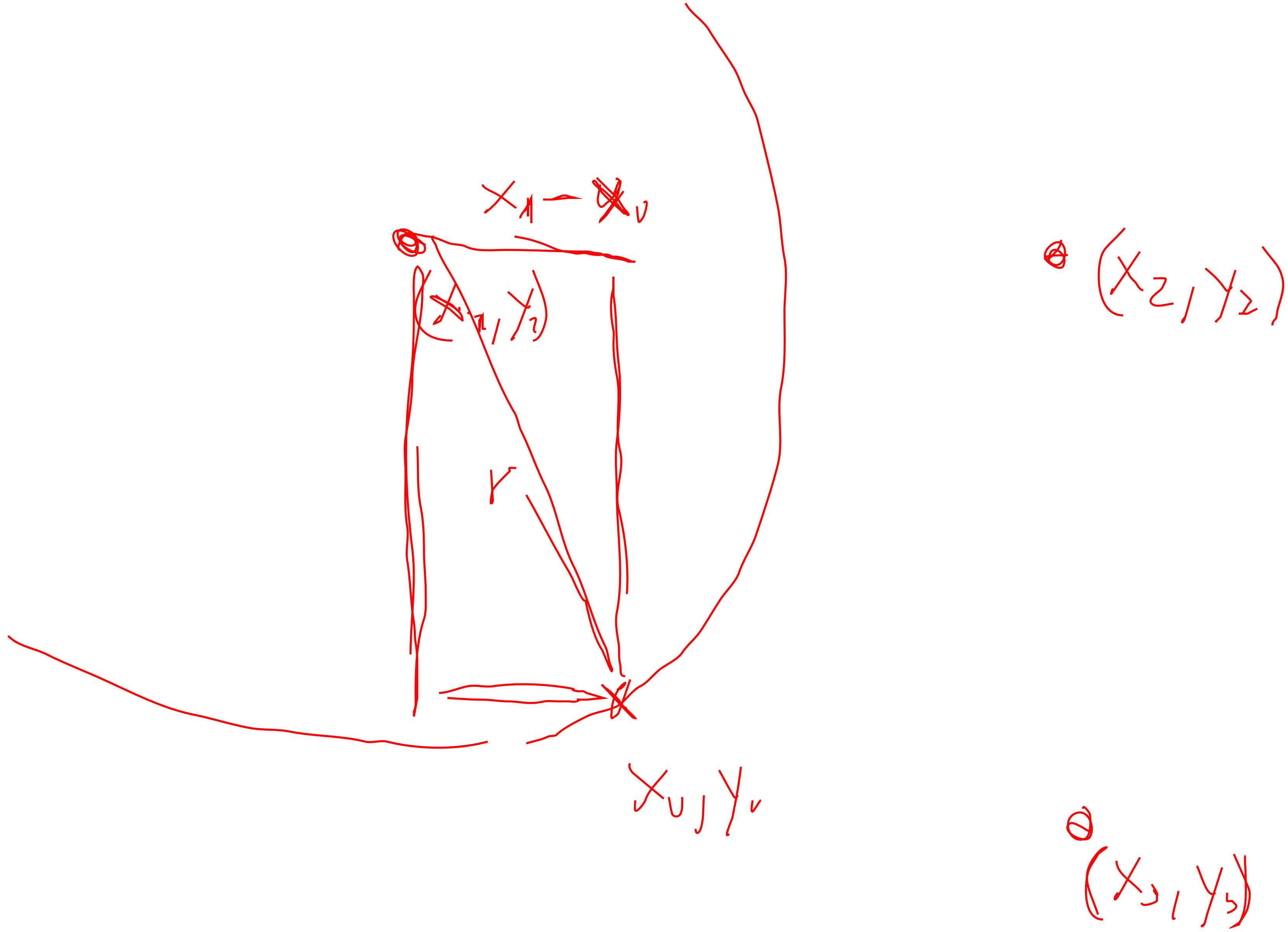
$$(x_i - x_u)^2 + (y_i - y_u)^2 = r_i^2 \text{ for } i = 1, \ldots, 3$$

- **Transformation**

$$(x_1 - x_u)^2 - (x_3 - x_u)^2 + (y_1 - y_u)^2 - (y_3 - y_u)^2 = r_1^2 - r_3^2$$
$$(x_2 - x_u)^2 - (x_2 - x_u)^2 + (y_2 - y_u)^2 - (y_2 - y_u)^2 = r_2^2 - r_3^2.$$

- **results in:**

$$2(x_3 - x_1)x_u + 2(y_3 - y_1)y_u = (r_1^2 - r_3^2) - (x_1^2 - x_3^2) - (y_1^2 - y_3^2)$$
$$2(x_3 - x_2)x_u + 2(y_3 - y_2)y_u = (r_2^2 - r_2^2) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2)$$



X1 − Xu
(X1, Y1)
r
Xu, Yu
(X2, Y2)
(X3, Y3)

Trilateration as a Linear System of Equations

- **Forming a system of equations**

$$\underbrace{2\begin{bmatrix} x_3 - x_1 & y_3 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{bmatrix}}_{A \;\; 2\times 2} \begin{bmatrix} x_u \\ y_u \end{bmatrix} = \underbrace{\begin{bmatrix} (r_1^2 - r_3^2) - (x_1^2 - x_3^2) - (y_1^2 - y_3^2) \\ (r_2^2 - r_2^2) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) \end{bmatrix}}_{2\times 1 \quad b}$$

- **Example:**
  - **$(x_1, y_1) = (2,1)$, $(x_2, y_2) = (5,4)$, $(x_3, y_3) = (8,2)$,**
  - **$r_1 = 10^{1/2}$, $r_2 = 2$, $r_3 = 3$**

$$2\begin{bmatrix} 6 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_u \\ y_u \end{bmatrix} = \begin{bmatrix} 64 \\ 22 \end{bmatrix}$$

$Ax = b$

$x = A^{-1} b$

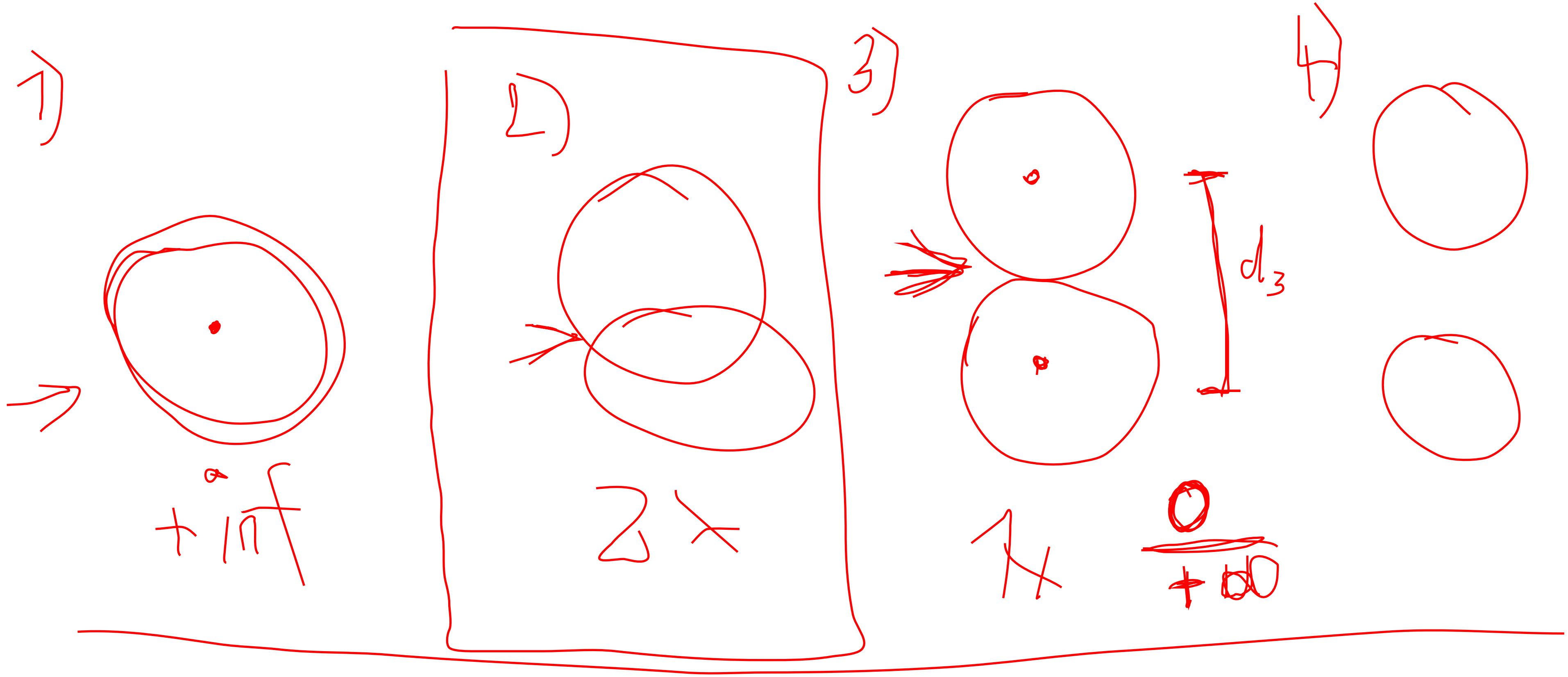
➔ **$(x_u,y_u) = (5,2)$**

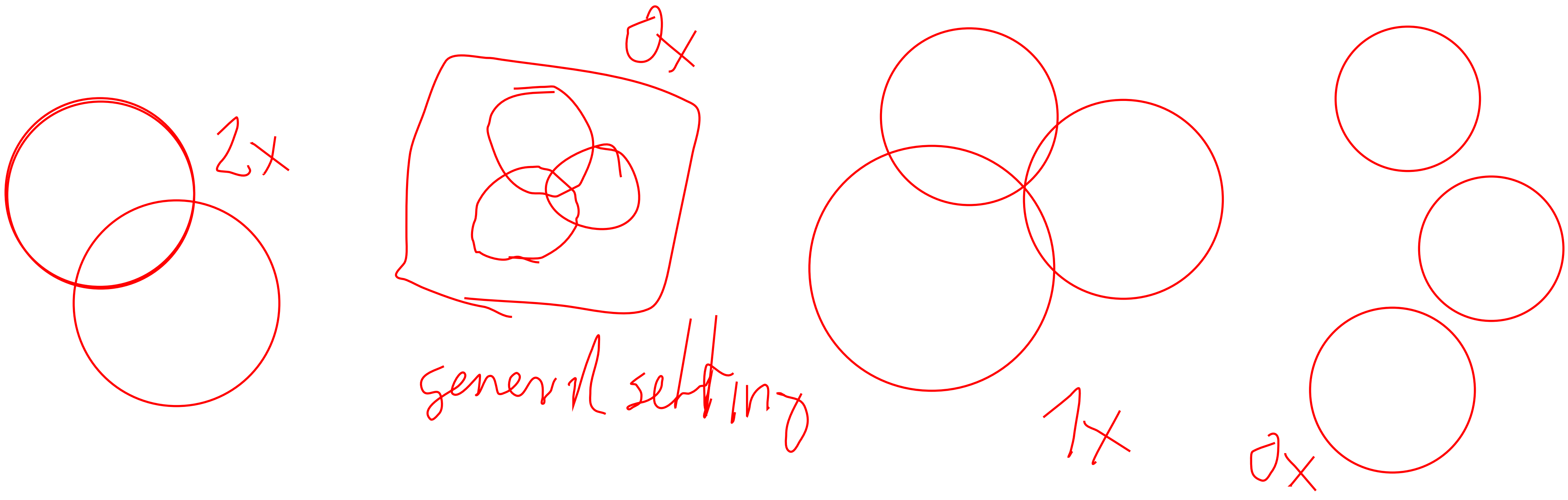
Trilateration as a Linear System of Equations

- **In three dimensions**
  - **Intersection of four spheres**

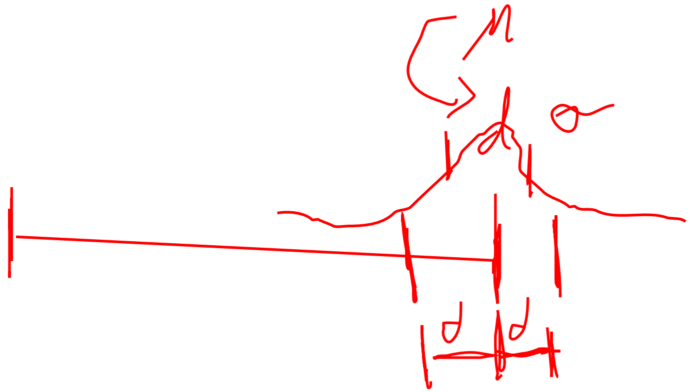
$$\underbrace{\begin{bmatrix} (d_1^2-d_4^2)-(x_1^2-x_4^2)-(y_1^2-y_4^2)-(z_1^2-z_4^2) \\ (d_2^2-d_4^2)-(x_2^2-x_4^2)-(y_2^2-y_4^2)-(z_2^2-z_4^2) \\ (d_3^2-d_4^2)-(x_3^2-x_4^2)-(y_3^2-y_4^2)-(z_3^2-z_4^2) \end{bmatrix}}_{\vec{b}} = 2 \underbrace{\begin{bmatrix} (x_4-x_1) & (y_4-y_1) & (z_4-z_1) \\ (x_4-x_2) & (y_4-y_2) & (z_4-z_2) \\ (x_4-x_3) & (y_4-y_3) & (z_4-z_3) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_{P1} \\ y_{P1} \\ z_{P1} \end{bmatrix}}_{\vec{x}}$$

- **Solve $\mathbf{Ax} = \mathbf{b}$ ➔ $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$**

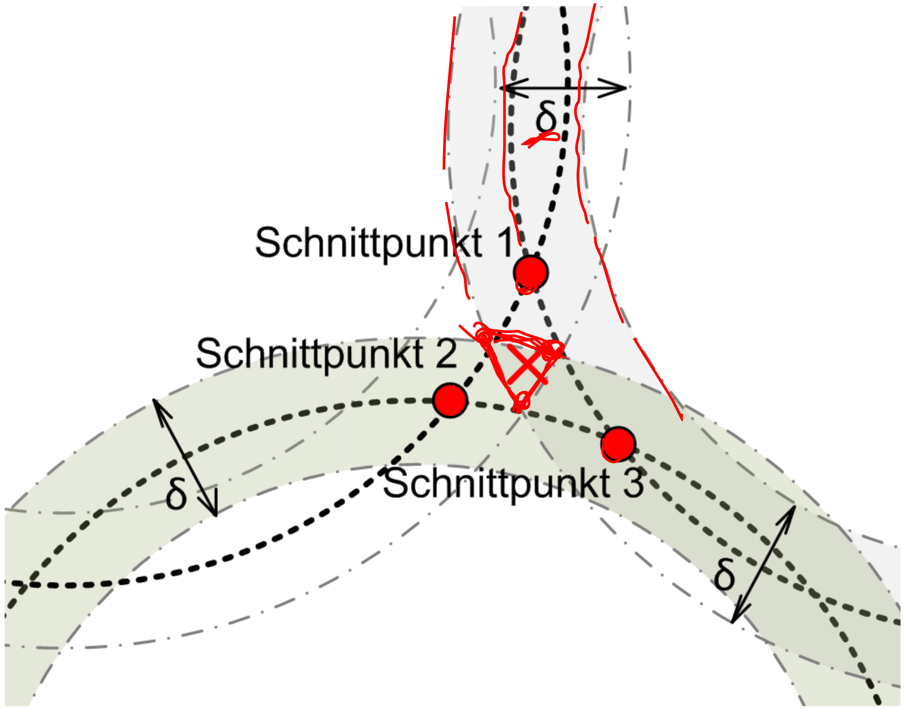

1)
+inf
2)
2x
3)
d_3
1x
$\frac{0}{+\infty}$
4)


2x
0x
general setting
1x
0x

Trilateration

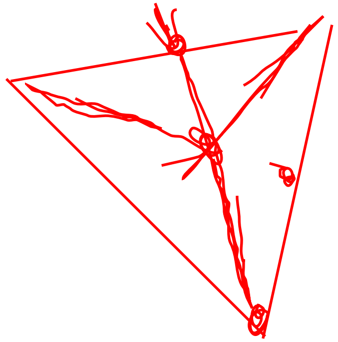


- **In case of measurement errors**



[F. Höflinger, 2013]

- **Averaging: e.g. centroid of triangle**

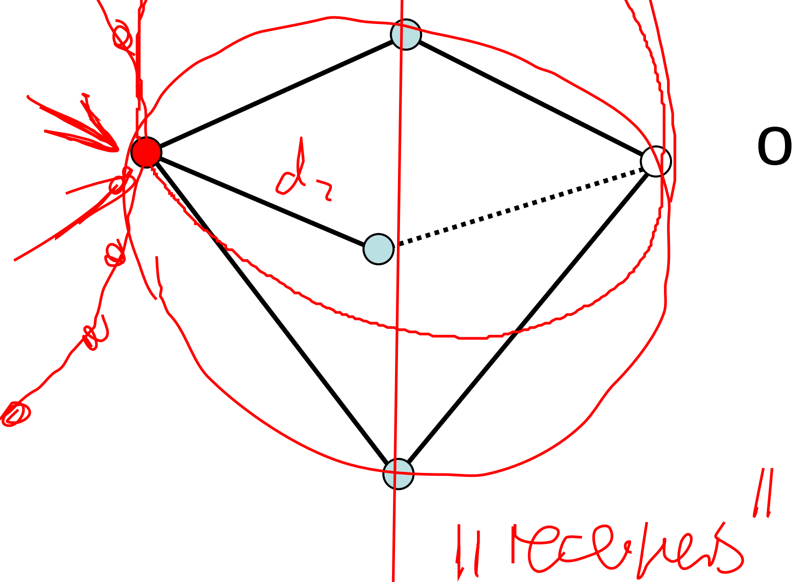


centroid

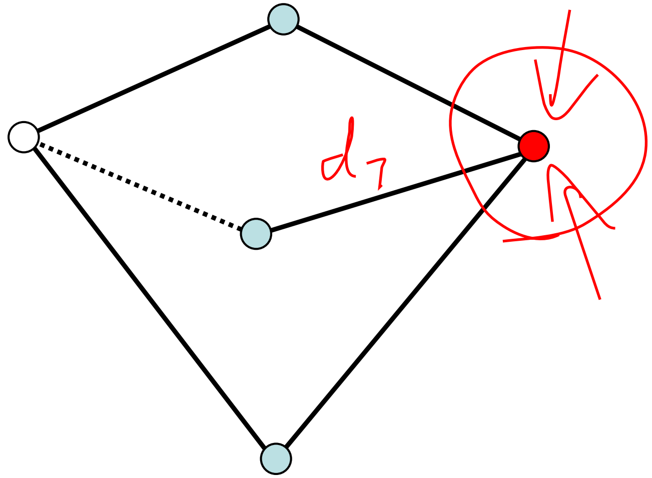
Trilateration

$$\frac{d_2}{d_1}$$

- Measurement errors
  - Small distance errors can lead to large position errors



or

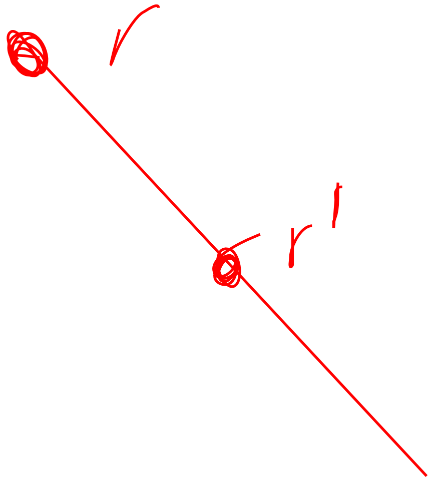


- flip ambiguity from noise

Multilateration with *absolute* distances

$Ax=b$

- **Multilateration (absolute distances): Calculate the intersection of *at least four* distance measurements**
  - **May be over-determined equation system: More equations than variables**
  - **"No solution" in case of measurement errors**
- **Minimize sum of quadratic residuals: *Least squares***
- **Vector notation**
  - **Solve** $(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$
  - **Matrix inverse by Gauss-Jordan elimination**

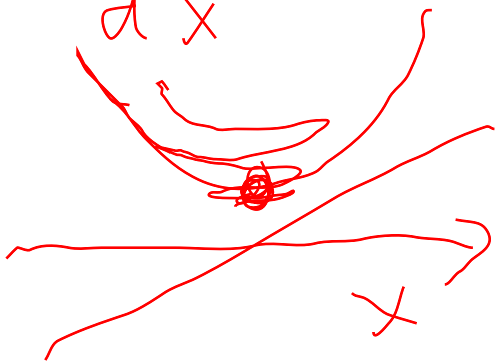


$|r - r'|$

$= \min r'$

$x \sim f(r')$

$$\frac{d\,|r - r'|}{dx}$$



$(r - r')^2$

$n = 2$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \\ \end{pmatrix} m$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$b = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}$$

$$Ax = b$$

$$(A^T A)x = A^T b$$

Linear least squares

Matrix elimination

$$x = (A^T A)^{-1}(A^T b)$$

$$A \rightarrow A^2 = A^T A$$

$n \times n$

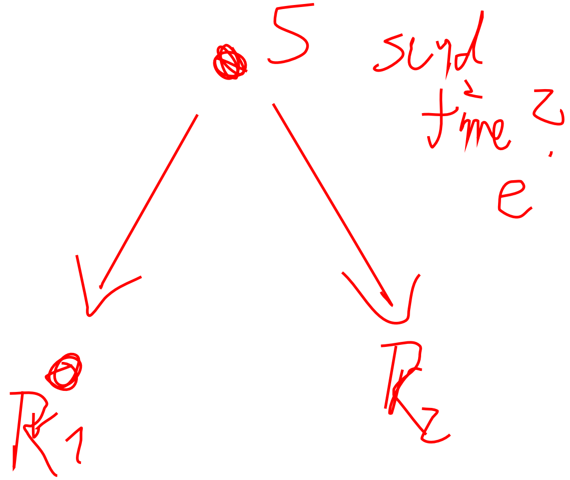
$$A \cdot A$$

$m \times n$ $\quad$ $m \times n$

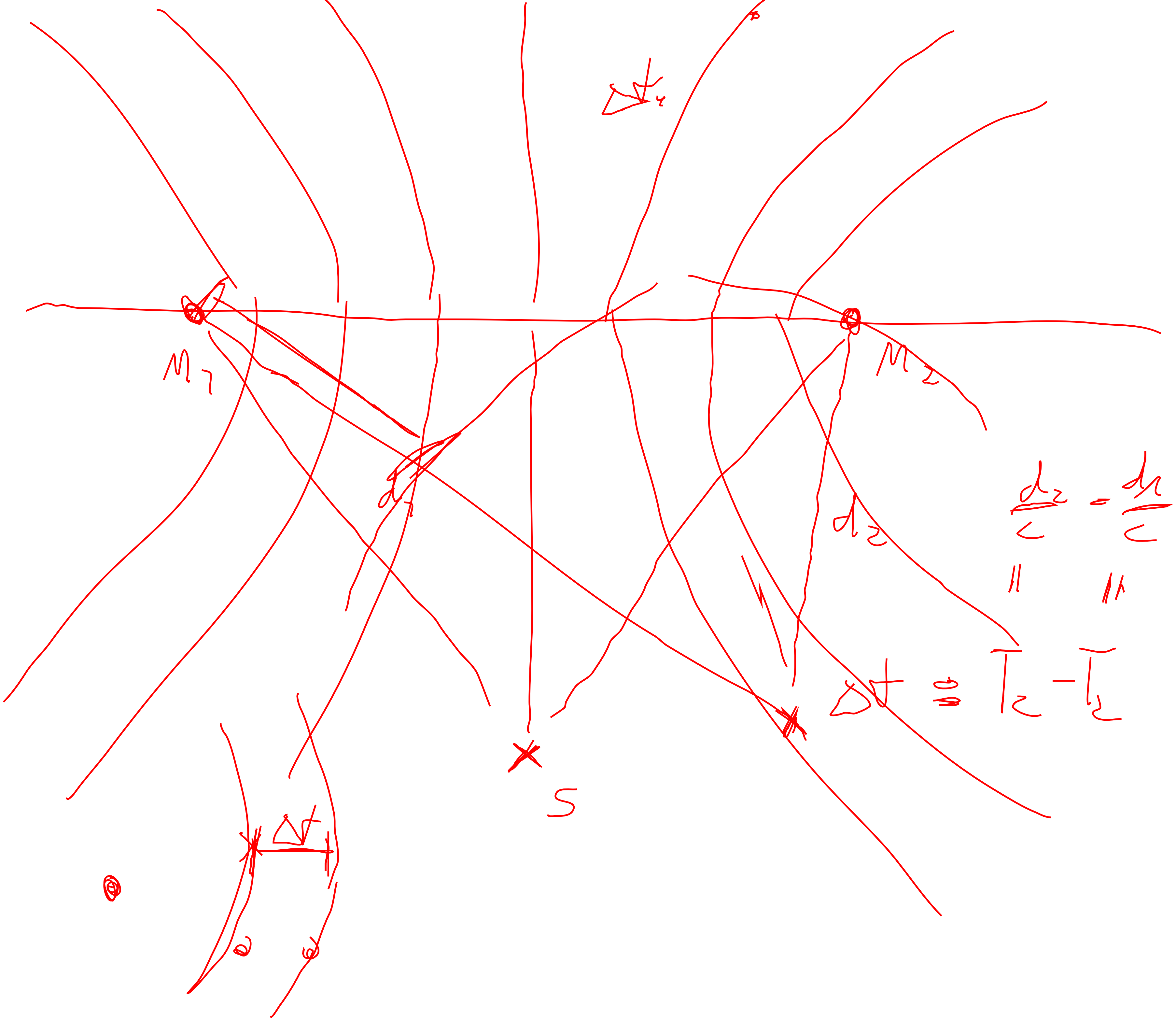
$m \neq n$

Multilateration with *relative* distances

- **Multilateration (relative): Calculate the intersection of *relative* distance measurements**
  - **Emission time e unknown!**
  - **Measure only reception times T_i, $i = 1, \ldots, n$**
  - **System of equations $T_i = e + \| r_i - s \| / c$**
  - **...for a signal traveling from s to receivers r_i**



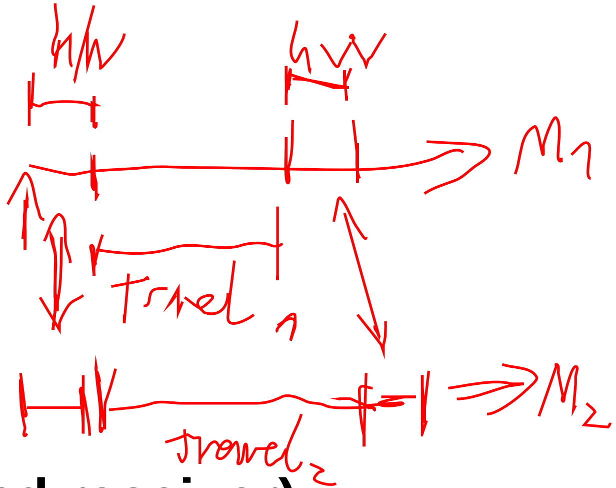
- **Subtract two absolute times T_i and T_j:**
  - $T_i - T_j = \| r_i - s \| / c - \| r_j - s \| / c =: \Delta t \quad (i, j = 1, \ldots, n)$
  - **System of hyperbolic equations**
  - **Time Difference of Arrival Δt, relative distance $\Delta d = c \Delta t$**



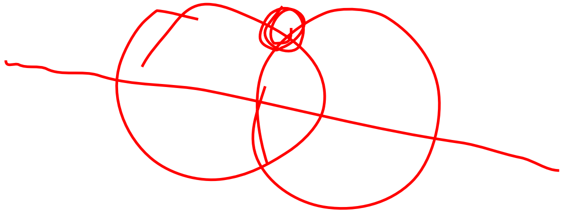
$$\frac{d_2}{c} - \frac{d_1}{c}$$

$$\Delta t = T_2 - T_1$$

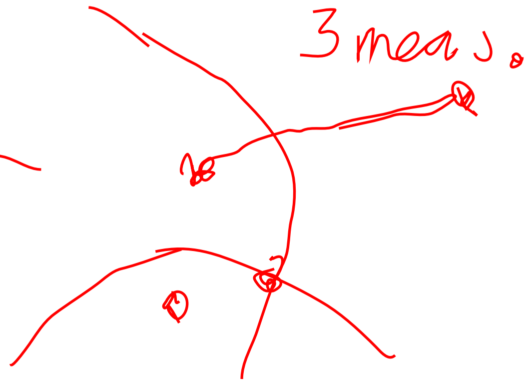
Multilateration with *relative* distances



- **Advantages**
  - **No cooperation of signal emitter**
  - **Hardware delays cancel out (both emitter and receiver)**
  - **Passive localization / natural signal sources**
- **Disadvantages**
  - **Larger number of unknown values: Position and time**
  - **Synchronization still (usually) required**



2 meas

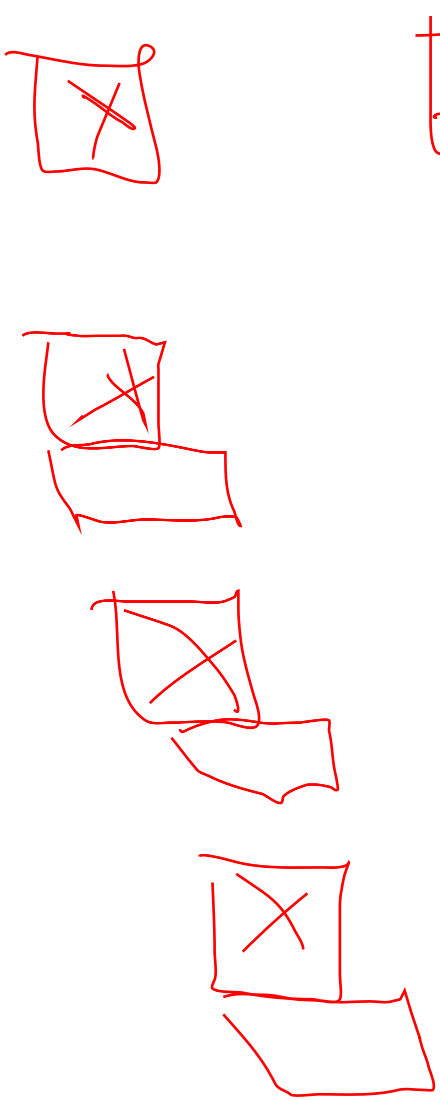


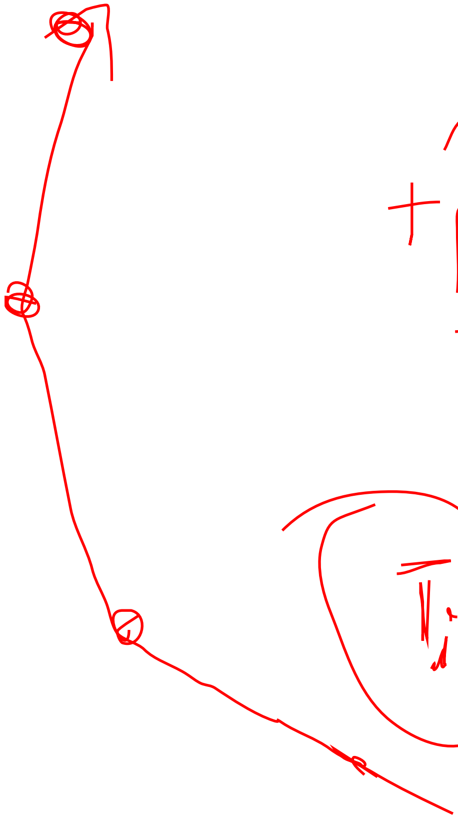
2D : 2 → 3

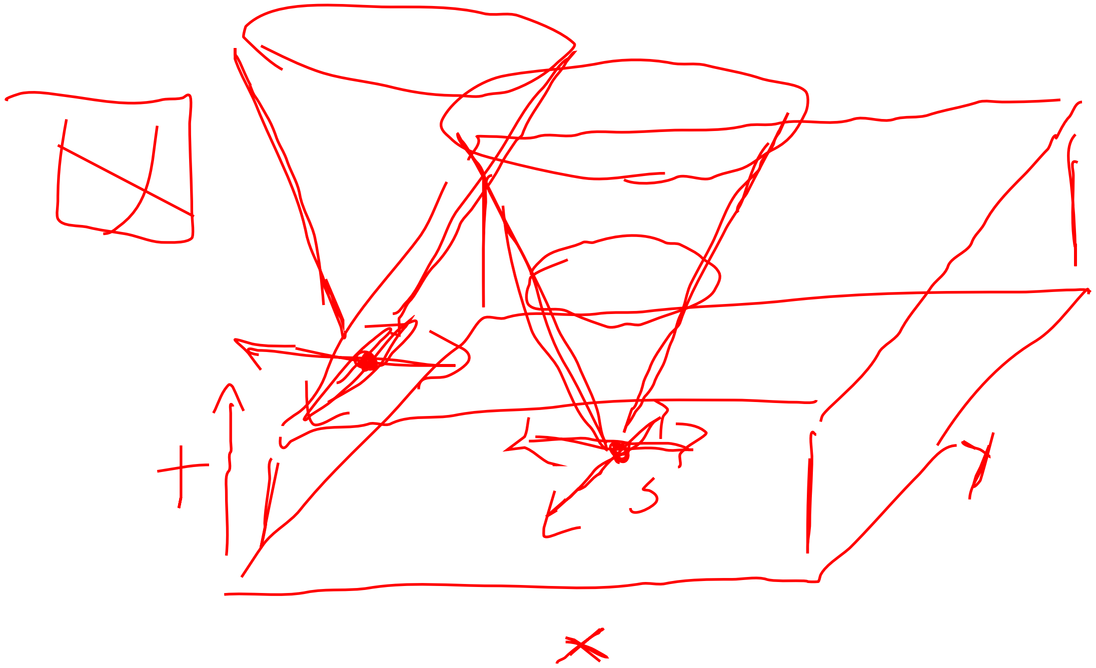
3D : 3 → 4

Anchor-free localization

- "Anchor-free localization":
  - Hyperbolic multilateration
  - Unknown emitters $\mathbf{s}_j$, and unknown receivers $\mathbf{r}_i$
- Advantages:
  - No need to measure receiver positions
  - Self-positioning by passive information from the surroundings
- Disadvantages:
  - Even larger number of unknown variables

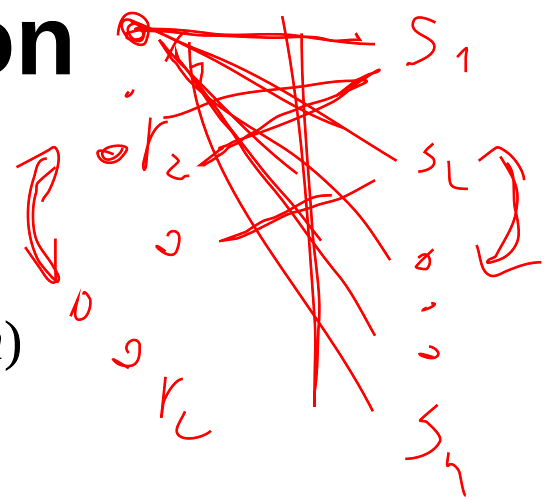




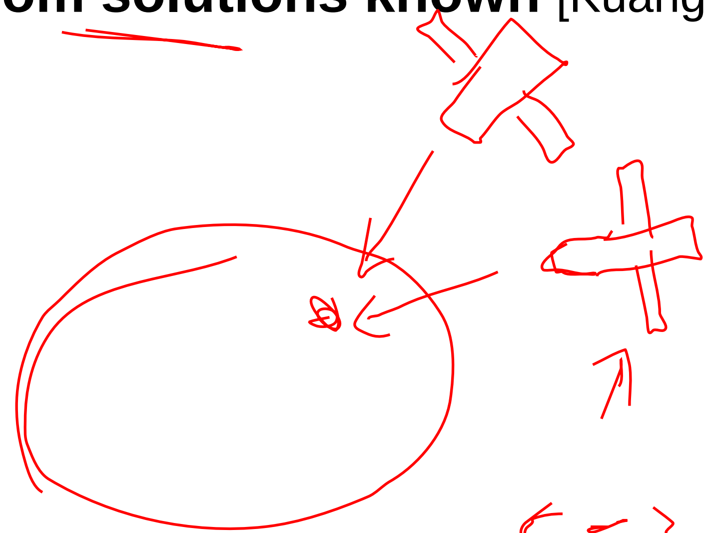


$$T_{ij} = e + \|r_i - s_j\| / c$$

Anchor-free localization



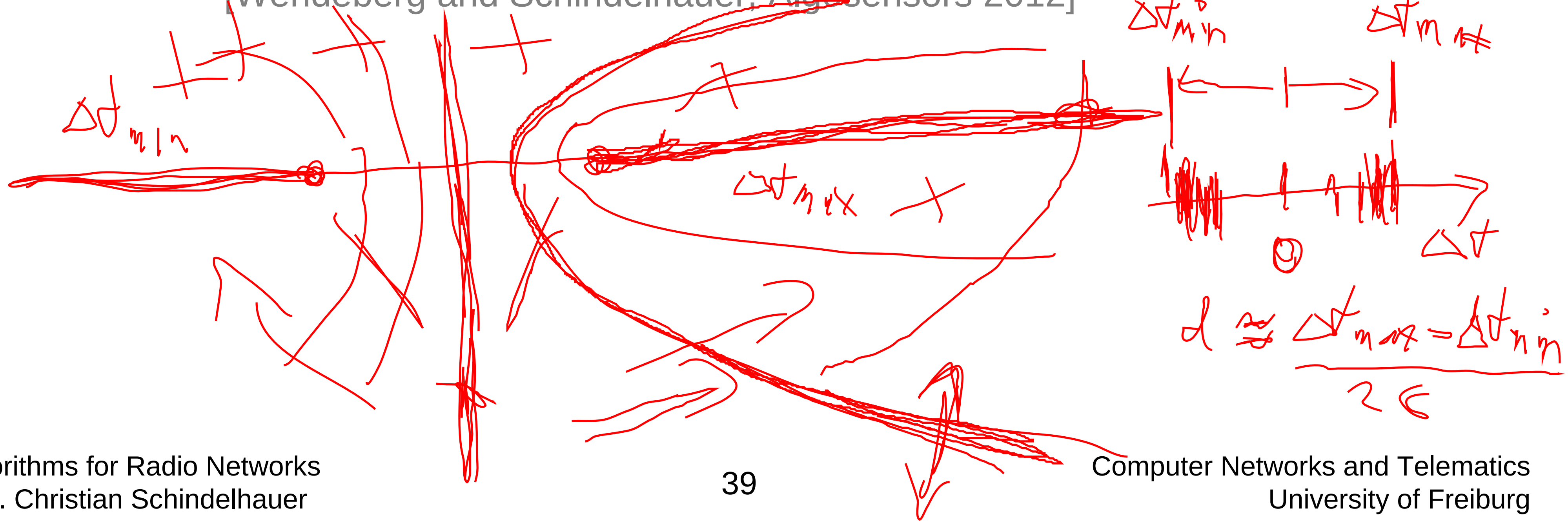
- **For absolute distances** d_{ik}**:**
  - **Solve** $\| \mathbf{r}_i - \mathbf{s}_k \| = d_{ik} \qquad (i, j = 1, \ldots, n\ ;\ \ k = 1, \ldots, m)$
  - **Problem of intersecting circles / spheres**
  - **Bipartite distance graph:** $G = \left(\{\mathbf{r}_i\}, \{\mathbf{s}_k\}, \{d(i, k)\}\right)$
  - **Minimum case closed-from solutions known** [Kuang, et al., ICASSP 2013]



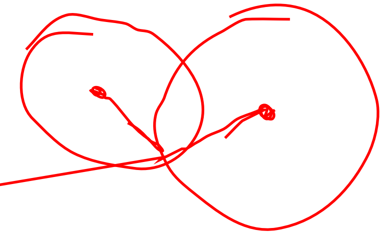
$c = 3 \cdot 10^8$

Anchor-free localization

- For *relative* distances $\Delta d_{ijk} = d_{ik} - d_{jk}$:
  - Solve $\| \mathbf{r}_i - \mathbf{s}_k \| - \| \mathbf{r}_j - \mathbf{s}_k \| = \Delta d_{ijk}$
  - Problem of intersecting hyperbolas / hyperboloids
  - Closed-form solutions only for larger problem sets
    [Pollefeys and Nister, ICASSP 2008], [Kuang and Åström, EUSIPCO 2013]
  - Minimum problem set: Iterative/recursive approximations
    [Wendeberg and Schindelhauer, Algosensors 2012]



Anchor-free localization



- **Degrees of freedom**

$T_{ik} = e_{ik} + \| \mathbf{r}_i - \mathbf{s}_k \| / \mathbf{c}$

(e_{ik}, $\mathbf{r}_i$, $\mathbf{s}_k$ unknown)

2D

⇒ expensive

m n

cheap

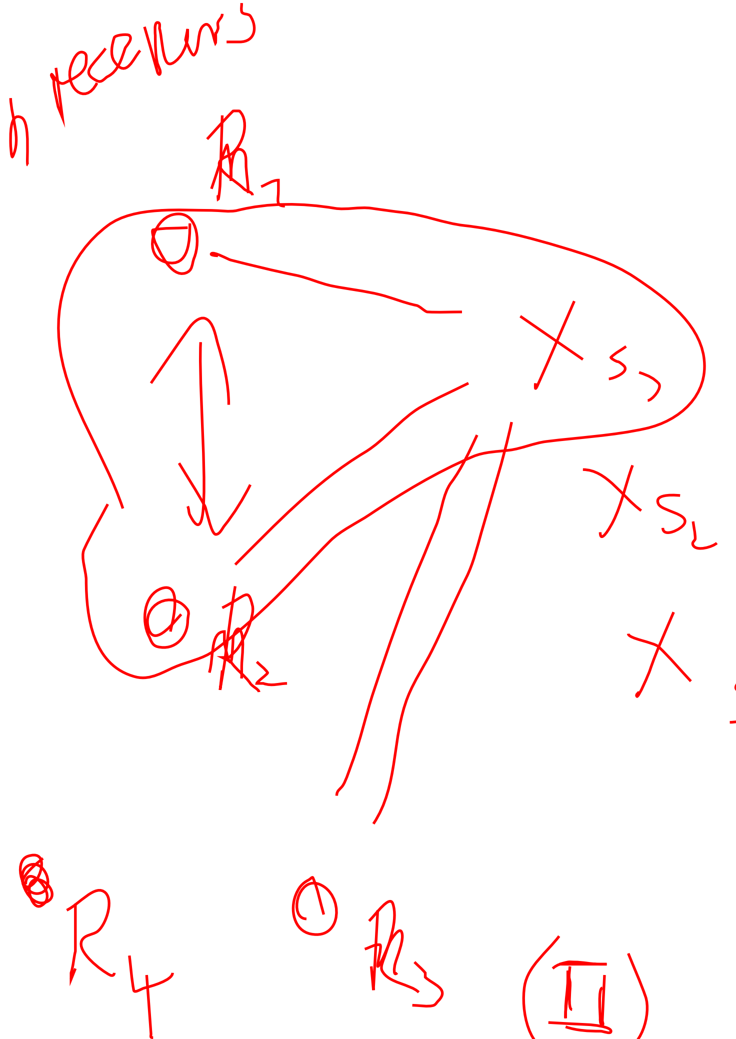
signal sources	receivers 1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	3	3	3	3	3	3	3	3
3	5	4	3	2	1	0	-1	-2
4	7	5	3	1	-1	-3	-5	-7
5	9	6	3	0	-3	-6	-9	-12
6	11	7	3	-1	-5	-9	-13	-17
7	13	8	3	-2	-7	-12	-17	-22
8	15	9	3	-3	-9	-15	-21	-27
9	17	10	3	-4	-11	-18	-25	-32
10	19	11	3	-5	-13	-21	-29	-37
11	21	12	3	-6	-15	-24	-33	-42
12	23	13	3	-7	-17	-27	-37	-47

signal sources	receivers 1	2	3	4	5	6	7	8
1	0	2	4	6	8	10	12	14
2	3	4	5	6	7	8	9	10
3	6	6	6	6	6	6	6	6
4	9	8	7	6	5	4	3	2
5	12	10	8	6	4	2	0	-2
6	15	12	9	6	3	0	-3	-6
7	18	14	10	6	2	-2	-6	-10
8	21	16	11	6	1	-4	-9	-14
9	24	18	12	6	0	-6	-12	-18
10	27	20	13	6	-1	-8	-15	-22
11	30	22	14	6	-2	-10	-18	-26
12	33	24	15	6	-3	-12	-21	-30

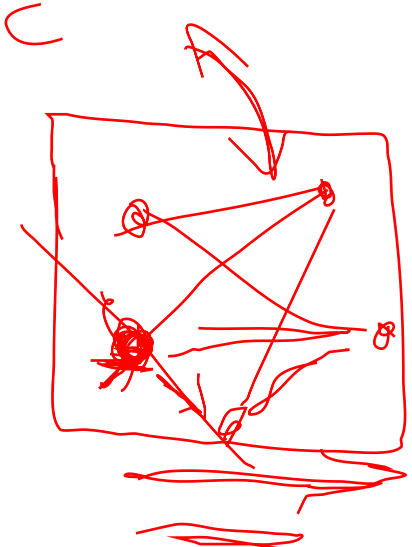
$$G_{2D} = \underbrace{2n + 3m}_{\text{receivers}} \underbrace{- nm - 3}_{\text{senders}}$$

$$G_{3D} = 3n + 4m - nm - 6$$

2)



$$e^{-\frac{\|R_i - S\|}{c}} = \tau_i$$



(x, y, α)

$n = 2$, $m = 1$ → (I) 7, (II) 5

(I)

$2n + 3m$

$n \cdot 2 + 2 \cdot m + m$

rec

(II) ⟺ $mn + 3$

$m \cdot n + 3$

Anchor-free localization

- **Minimum cases**

	2D	3D
general setting	**4 / 6**	**5 / 10** **6 / 7**
far-field setting	**3 / 3 (sync.)** **3 / 5 (unsync.)**	**4 / 6 (sync.)** **4 / 9 (unsync.)**

Minimum number of required receivers / emitters

ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG

Algorithms for Radio Networks

Localization

University of FreiburgTechnical Faculty
Computer Networks and Telematics
Prof. Christian Schindelhauer

CoNe Freiburg

IIF INSTITUT FÜR INFORMATIK FREIBURG