Minimal Solvers for Unsynchronized TDOA Sensor Network Calibration

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Abstract. We present two novel approaches for the problem of self-calibration of network nodes using only TDOA when both receivers and transmitters are unsynchronized. We consider the previously unsolved minimum problem of far field localization in three dimensions, which is to locate four receivers by the signals of nine unknown transmitters, for which we assume that they originate from far away. The first approach uses that the time differences between four receivers characterize an ellipsoid. The second approach uses linear algebra techniques on the matrix of unsynchronized TDOA measurements. This approach is easily extended to more than four receivers and nine transmitters. In extensive experiments, the algorithms are shown to be robust to moderate Gaussian measurement noise and the far field assumption is reasonable if the distance between transmitters and receivers is at least four times the distance between the receivers. In an indoor experiment using sound we reconstruct the microphone positions up to a mean error of 5 cm.

1 Introduction

In this paper we study the problem of node localization using only Unsynchronized Time difference Of Arrival (UTOA) measurements between nodes, where either receivers or transmitters are far away from the other group. The problem arises naturally in microphone arrays for audio sensing. Is it possible to calculate both multiple microphone positions as well as the timings and directions of the sound sources, if the microphones are unsynchronized, i.e. do not use the same clock, just from sounds emanating from far away at unknown locations and times? An example application could be to locate several cell phones just by environmental sounds, where cell phone positions and sound directions are to be recovered without synchronizing the phones first.

1.1 Related Work

Although time of arrival (TOA) and time difference of arrival (TDOA) problems have been studied extensively in the literature in the form of localization of e.g. a sound source using a calibrated array, see e.g. [4, 6–8], the problem of calibration of a sensor array from only measurements, i.e. the node localization problem, has received less attention.

In [21] and refined in [12] a far field approximation was utilized to solve the TOA and TDOA case, with the minimal number of four receivers and six unsynchronized far field transmitters in 3D. Under the assumption that signals and receivers are distributed in the unit disk, the distance between receivers can be approximated by evaluation of the range of time differences [3, 16, 18] or by statistical analysis of their distribution [10, 20], although these approaches depend on the availability of a large number of signals. Calibration of TOA networks using only measurements has been studied in [14, 19], where solutions to the minimal cases of three transmitters and three receivers in the plane, or six transmitters and four receivers in 3D are given. Calibration of TDOA networks is studied in [17] and further improved upon in [13], where the non-minimal case of eight transmitters and five receivers is solved. In [2, 23] a TDOA setup is used for indoor navigation based on non-linear optimization, but the methods can get stuck in local minima and are dependent on initialization.

The problem of node localization using only UTOA measurements from unsynchronized receivers and transmitters in a far field setting has been considered in [5], however the approach requires at least five receivers, which is more than the minimum case. Minimal algorithms are of importance in RANSAC schemes [9] to weed out outliers in noisy data which is a common problem in TOA/TDOA/UTOA applications. The problem has been addressed in a different manner estimating ellipse coefficients in [22], but no analysis of degenerate cases has been done and the algorithm is only described for the planar case.

In this paper we expand on previous work and propose two novel algorithms for parameter estimation of a receiver array, the Ellipsoid method in 3D and the Matrix Factorization method for UTOA measurements, that both consider the minimum case of four receivers and nine transmitters in three dimensions. We compare the methods on simulated and real data where we demonstrate their numerical stability. The methods are also evaluated on overdetermined cases using more than four receivers and nine transmitters.

2 Problem Setting

In the following treatment, we make no difference between real and virtual transmitters. Assume that the transmitters are stationary at position $\mathbf{b}_j \in \mathbb{R}^3$, $j = 1, \ldots, k$ and that the receivers are at positions $\mathbf{r}_i \in \mathbb{R}^3,\ i = 1, \ldots, m$. By measuring how long time the signals take to reach the receiver and knowing the speed of the signals, distances $\delta_{ij} = \|\mathbf{r}_i - \mathbf{b}_j\|$ can be measured, $\|\cdot\|$ denoting the Euclidean norm. These are TOA measurements.

When neither receivers or transmitters are synchronized, for instance external sound sources recorded on different computers, the measurements will be of the form $\delta_{ij} = \|\mathbf{r}_i - \mathbf{b}_j\| + f_i + \tilde{g}_j$ where $f_i, \tilde{g}_j$ are unknown offsets for receivers and transmitters respectively. We denote measurements of this kind Unsynchronized

Time difference Of Arrival (UTOA) measurements. Furthermore, if the transmitters are so far from the receivers that a transmitter can be considered to have a common direction to the receivers, the measurements can be approximated by

$$\delta_{ij} = \|\mathbf{r}_i - \mathbf{b}_j\| + f_i + \tilde{g}_j \approx \|\mathbf{r}_1 - \mathbf{b}_j\| + (\mathbf{r}_i - \mathbf{r}_1)^T \mathbf{n}_j + f_i + \tilde{g}_j = \mathbf{r}_i^T \mathbf{n}_j + \bar{g}_j + f_i + \tilde{g}_j \quad (1)$$

where $\bar{g}_j = \|\mathbf{r}_1 - \mathbf{b}_j\| - \mathbf{r}_1^T \mathbf{n}_j$ and $\mathbf{n}_j$ is the direction of unit length from transmitter j to the receivers. By setting $g_j = \bar{g}_j + \tilde{g}_j$ we get the far field approximation

$$\delta_{ij} \approx \mathbf{r}_i^T \mathbf{n}_j + f_i + g_j .$$

When the approximation is good, we will call δ_{ij} Far Field UTOA (FFUTOA) measurements.

2.1 The FFUTOA calibration problem

We assume that (i) the speed of signals v is known, and thus all time measurements are transformed to distances by multiplication by v and (ii) receivers can distinguish which TOA signal comes from which sender. This can be done in practice by e.g. separating the signals temporally or by frequency.

Problem 1. Given mk FFUTOA measurements $\delta_{ij} \in \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, k$, taken from m receivers and k transmitters, estimate receiver positions $\mathbf{r}_i \in \mathbb{R}^3$, directions $\mathbf{n}_j \in \mathbb{R}^3$ from transmitter j to receivers, receiver and transmitter offsets $f_i \in \mathbb{R}$, $g_j \in \mathbb{R}$ such that

$$\delta_{ij} = \mathbf{r}_i^T \mathbf{n}_j + f_i + g_j \ , \quad \text{and} \quad \|\mathbf{n}_j\| = 1 \ . \quad (2)$$

Note that the problem is symmetric in receivers and transmitters, i.e. if each receiver instead could be viewed as having a common direction to all transmitters, the same problem can be solved for transmitter positions and receiver directions. We denote $\mathbf{f} = [f_1, \ldots, f_m]^T$, $\mathbf{g} = [g_1, \ldots, g_k]$, $\mathbf{r} = [\mathbf{r}_1, \ldots, \mathbf{r}_m]$ and $\mathbf{n} = [\mathbf{n}_1, \ldots, \mathbf{n}_k]$.

The problem of determining full transmitter positions $\mathbf{b}_j$ instead of directions $\mathbf{n}_j$, see (1), seems harder than using the far field approximation as in Problem 1. The measurements are now bilinearly dependent on $\mathbf{r}_i$ and $\mathbf{n}_j$. Algorithms that explicitly consider the far field assumption are also required, as the problem of determining general positions of transmitters when the far field approximation is in effect, is an ill-conditioned problem.

We denote the problem as minimal if the number of solutions for generic distance measurements δ_{ij} is finite and positive, disregarding solutions that are the same up to gauge freedom.

2.2 Gauge freedom

The unknown parameters $(\mathbf{r}, \mathbf{n}, \mathbf{f}, \mathbf{g})$ have certain degrees of freedom that does not change the measurements, called gauge freedom. Any translation $\mathbf{t}$, rotation

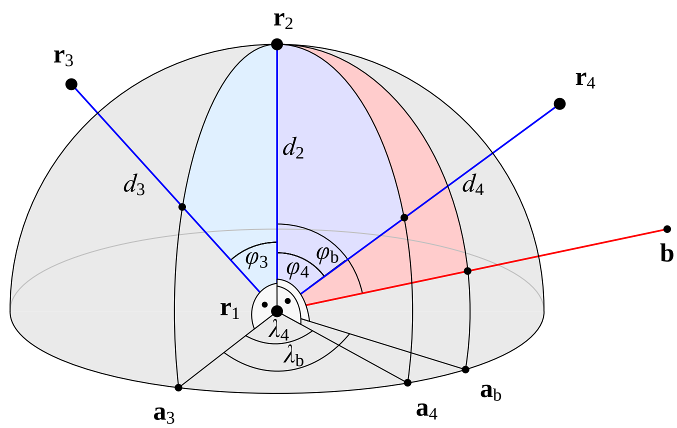


Fig. 1. Scheme of the Ellipsoid method. Three distances d_2, d_3, d_4, and three angles φ_3, φ_4, and θ define a tetrahedron of four receivers $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$. Transmitter $\mathbf{b}$ is assumed to be far away from the receivers. Its signal arrives from the angles φ_b, λ_b.

matrix $\mathbf{R}$ and offset change K can be applied to the solution according to

$$\mathbf{r}_{i,\mathrm{trans}} = \mathbf{r}_i + \mathbf{t},\ g_{j,\mathrm{trans}} = g_j - \mathbf{t}^T\mathbf{n}_j$$
$$\mathbf{r}_{i,\mathrm{rot}} = \mathbf{R}\mathbf{r}_i,\ \mathbf{n}_{j,\mathrm{rot}} = \mathbf{R}\mathbf{n}_j$$
$$f_{i,\mathrm{offs}} = f_i + K,\ g_{j,\mathrm{offs}} = g_j - K$$

without changing the measurements δ_{ij}. Thus, we can only hope to reconstruct the unknowns up to these seven degrees of freedom.

3 The Ellipsoid Method in Three-Dimensional Space

We propose the Ellipsoid TDOA method which solves the FFUTOA calibration problem for four receivers using at least nine transmitters. The time differences of signals from distant emitters form an ellipsoid which characterizes the distances and angles between four receivers. An elegant representation can be derived from the knowledge that an ellipsoid corresponds to a covariance matrix. Once this covariance matrix is known, one can extract the parameters that generate the ellipsoid from the matrix, i.e. the configuration of four receivers.

3.1 Definition of the Covariance Ellipsoid

A rigid tetrahedron of four receivers is defined by three distances $d_2 = \|\mathbf{r}_1 - \mathbf{r}_2\|$, $d_3 = \|\mathbf{r}_1 - \mathbf{r}_3\|$, $d_4 = \|\mathbf{r}_1 - \mathbf{r}_4\|$, two height angles $\varphi_3 = \angle_{\mathbf{r}_2\mathbf{r}_1\mathbf{r}_3}$, $\varphi_4 = \angle_{\mathbf{r}_2\mathbf{r}_1\mathbf{r}_4}$, and the azimuth angle $\lambda_4 = \angle_{\mathbf{a}_3\mathbf{r}_1\mathbf{a}_4}$, see Fig. 1. Furthermore we define $\theta = \angle_{\mathbf{r}_3\mathbf{r}_1\mathbf{r}_4}$.

A signal arrives from the angles $\varphi_\mathrm{b} = \angle_{\mathbf{r}_2\mathbf{r}_1\mathbf{b}}$ and $\lambda_\mathrm{b} = \angle_{\mathbf{a}_3\mathbf{r}_1\mathbf{a}_\mathrm{b}}$, uniquely determining the direction. The signal angles with respect to two receivers are

$\gamma_2 = \angle_{\mathbf{r}_2\mathbf{r}_1\mathbf{b}}$, $\gamma_3 = \angle_{\mathbf{r}_3\mathbf{r}_1\mathbf{b}}$, and $\gamma_4 = \angle_{\mathbf{r}_4\mathbf{r}_1\mathbf{b}}$. Omitting the signal index, these angles are defined by the UTOA measures according to the cosine law as

$$\begin{aligned} x = \delta_1 - \delta_2 = d_2 \cos(\gamma_2)\ , \quad & y = \delta_1 - \delta_3 = d_3 \cos(\gamma_3)\ , \\ \text{and} \quad & z = \delta_1 - \delta_4 = d_4 \cos(\gamma_4)\ . \end{aligned} \tag{3}$$

The auxiliary points $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_\mathrm{b}$ are projections of $\mathbf{r}_3$, $\mathbf{r}_4$, and $\mathbf{b}$ respectively, onto the plane orthogonal to $\mathbf{r}_1 - \mathbf{r}_2$ through $\mathbf{r}_1$.

In the following we derive the covariance matrix for time differences in the Eqns. (3) assuming uniform signal source positions. This matrix characterizes a covariance ellipsoid, [15], describing the ellipsoid which the time differences reside on. If this matrix is known, the distances and angles between the receivers can be directly read from the matrix. We state the following definition.

Definition 1. *The $\mathbf{\Sigma}$-ellipsoid for covariance matrix $\mathbf{\Sigma}$ is the ellipsoid with center $\boldsymbol{\mu}$ where for all points $\mathbf{x}$ holds*

$$d_{Mah}(\mathbf{x}, \boldsymbol{\mu}, \mathbf{\Sigma}) \ = \ \sqrt{(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})} \ = \ 1\ .$$

The metric $d_{\mathrm{Mah}}(\mathbf{x}, \boldsymbol{\mu}, \mathbf{\Sigma})$ is the Mahalanobis distance. For $\mathbf{\Sigma}$-ellipsoids the following holds.

Lemma 1. *The covariance of points uniformly distributed over a $\mathbf{\Sigma}$-ellipsoid in $\mathbb{R}^3$ is $\hat{\mathbf{\Sigma}} = \frac{1}{3}\mathbf{\Sigma}$. In the two-dimensional case the covariance is $\hat{\mathbf{\Sigma}} = \frac{1}{2}\mathbf{\Sigma}$.*

Lemma 1 can be verified by integration over all points of the $\mathbf{\Sigma}$-ellipsoid and calculating the covariance. Given the definition of the covariance ellipsoid we propose the following theorem.

Theorem 1. *The time differences (x, y, z) of distant signals arriving at four receivers $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ in space $\mathbb{R}^3$ form a $3\hat{\mathbf{\Sigma}}$-ellipsoid with covariance matrix*

$$\hat{\mathbf{\Sigma}} \ = \ \frac{1}{3} \begin{pmatrix} d_2^2 & d_2 d_3 \cos(\varphi_3) & d_2 d_4 \cos(\varphi_4) \\ d_2 d_3 \cos(\varphi_3) & d_3^2 & d_3 d_4 \cos(\theta) \\ d_2 d_4 \cos(\varphi_4) & d_3 d_4 \cos(\theta) & d_4^2 \end{pmatrix}\ .$$

Proof: The proof is directly based on the definition of a covariance ellipsoid. The first thing to show is that the matrix $\hat{\mathbf{\Sigma}}$ is actually a covariance matrix, therefore is positive semi-definite. For simplicity we assume that the receivers are synchronized, therefore the mean $\boldsymbol{\mu}$ is zero. In case they are not, synchronize the receivers by regression as described in the next Section 3.2.

Now, consider the continuous distribution of synchronized time differences over uniformly distributed directions of origin. Such a uniform distribution of signal origins $\hat{\mathbf{b}}$ in space $\mathbb{R}^3$ can be created by points

$$\hat{\mathbf{b}} \ = \ R \cdot \big(r\cos(\lambda),\ r\sin(\lambda),\ \ell\big)^T\ ,$$

where $\lambda \in [0, 2\pi]$ and $\ell \in [-1, 1]$ are uniformly independently distributed random variables, and $r = \sqrt{1 - \ell^2}$. The density function of the distribution is $h(\lambda, \ell) =$

$g(\lambda)f(\ell) = \frac{1}{4\pi}$. Without loss of generality, the tetrahedron is aligned such that $\mathbf{r}_1$ is the origin, $\mathbf{r}_2$ is parallel to the $\hat{z}$-axis, and $\mathbf{r}_3$ resides on the $\hat{x}/\hat{z}$-plane. Assuming that the sphere is large, i.e. the signals $\hat{\mathbf{b}}$ originate from far away, the angles of the signals are

$$\lambda_\mathrm{b} = \lambda \qquad \text{and} \qquad \cos(\varphi_\mathrm{b}) = \ell \; . \tag{4}$$

By using spherical trigonometry and the Eqns. (3) we calculate the time differences $\hat{\mathbf{x}} = [x, y, z]^T$ with respect to the tetrahedron angles as follows

$$\begin{aligned} x &= d_2 \cos(\gamma_2) = d_2 \Big(\cos(\varphi_\mathrm{b}) \Big) \\ y &= d_3 \cos(\gamma_3) = d_3 \Big(\cos(\varphi_3) \cos(\varphi_\mathrm{b}) + \sin(\varphi_3) \sin(\varphi_\mathrm{b}) \cos(\lambda_\mathrm{b}) \Big) \\ z &= d_4 \cos(\gamma_4) = d_4 \Big(\cos(\varphi_4) \cos(\varphi_\mathrm{b}) + \sin(\varphi_4) \sin(\varphi_\mathrm{b}) \cos(\lambda_\mathrm{b} - \lambda_4) \Big) \; . \end{aligned} \tag{5}$$

Note that the angles γ_2, γ_3, γ_4, are not uniformly distributed in the three-dimensional case, in contrast to the planar case. We express θ as

$$\cos(\theta) = \sin(\varphi_3) \sin(\varphi_4) \cos(\lambda_4) + \cos(\varphi_3) \cos(\varphi_4) \; . \tag{6}$$

Using the uniform distribution of signals (4) and the time differences $\hat{\mathbf{x}}$ from Eqns. (5) that follow, we show by integration that the time differences characterize a covariance matrix as stated in Theorem 1:

$$\begin{aligned} \hat{\mathbf{\Sigma}} &= \int_0^{2\pi} \int_{-1}^{1} \hat{\mathbf{x}} \, \hat{\mathbf{x}}^T \, h(\gamma, \ell) \, \mathrm{d}\ell \, \mathrm{d}\lambda = h(\gamma, \ell) \int_0^{2\pi} \int_{-1}^{1} \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \mathrm{d}\ell \, \mathrm{d}\lambda \\ &\overset{(4)-(6)}{=} \frac{1}{3} \begin{pmatrix} d_2^2 & d_2 d_3 \cos(\varphi_3) & d_2 d_4 \cos(\varphi_4) \\ d_2 d_3 \cos(\varphi_3) & d_3^2 & d_3 d_4 \cos(\theta) \\ d_2 d_4 \cos(\varphi_4) & d_3 d_4 \cos(\theta) & d_4^2 \end{pmatrix} \; . \end{aligned} \tag{7}$$

Due to the quadratic form is $\hat{\mathbf{\Sigma}}$ positive semidefinite. Furthermore, the matrix is definite, which follows from the fact that the time differences are bounded.

The next step is to verify that the time differences are actually characterized by the matrix. The distribution of signal directions $(\lambda_\mathrm{b}, \varphi_\mathrm{b})$ is irrelevant for this step, and for application of the algorithm. However, as the points $\hat{\mathbf{b}}$ in Eq. (4) cover the complete sphere, all signal directions are considered. Calculating the Mahalanobis distance by inserting $\hat{\mathbf{x}}$ and $\hat{\mathbf{\Sigma}}$ yields

$$d_\mathrm{Mah}\big(\hat{\mathbf{x}}, \vec{0}, 3\hat{\mathbf{\Sigma}}\big) = \sqrt{\hat{\mathbf{x}}^T \big(3\hat{\mathbf{\Sigma}}\big)^{-1} \hat{\mathbf{x}}} = 1 \; ,$$

revealing that all time difference points have constant Mahalanobis distance from the origin, therefore reside on an ellipsoid, which is according to Lemma 1 the $3\hat{\mathbf{\Sigma}}$-ellipsoid. □

3.2 Transformation of the Covariance Matrix

We now describe the transformation of parameters from a regression polynomial to the parameters of the covariance matrix. Under the assumption of a zero-mean ellipsoid, i.e. the receivers are synchronized, an ellipsoid is described by a polynomial equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 1 \; . \tag{8}$$

Regression of at least $m \geq 6$ signals in the equation system

$$\underbrace{\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & x_1z_1 & y_1z_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_m^2 & y_m^2 & z_m^2 & x_my_m & x_mz_m & y_mz_m \end{pmatrix}}_{\mathbf{Q}} \underbrace{(a, b, c, d, e, f)^T}_{\mathbf{u}} \;=\; \vec{1}$$

and solving a least squares scheme for $\mathbf{u} = \left(\mathbf{Q}^T\mathbf{Q}\right)^{-1}\left(\mathbf{Q}^T\vec{1}\right)$ yields ellipsoid parameters a to f.

An ellipsoid in space $\mathbb{R}^3$ can be represented by the matrix form $\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x} \;=\; 1$, where $\mathbf{x} = [x, y, z]^T$ is a vector and $\mathbf{\Sigma}$ is a symmetric positive definite matrix

$$\mathbf{\Sigma} \;=\; \begin{pmatrix} \sigma_1^2 & \omega_1 & \omega_2 \\ \omega_1 & \sigma_2^2 & \omega_3 \\ \omega_2 & \omega_3 & \sigma_3^2 \end{pmatrix} \; .$$

Substitution and conversion of the parameter set yields the parameters of the covariance matrix

$$\begin{aligned} \sigma_1^2 &= (f^2 - 4bc) \;/\; Z & \qquad \omega_1 &= (2cd - ef) \;/\; Z \\ \sigma_2^2 &= (e^2 - 4ac) \;/\; Z & \qquad \omega_2 &= (2be - df) \;/\; Z \\ \sigma_3^2 &= (d^2 - 4ab) \;/\; Z & \qquad \omega_3 &= (2af - de) \;/\; Z \end{aligned} \tag{9}$$

where $Z = be^2 + cd^2 + af^2 - 4abc - def$.

In case the receivers are not synchronized, the ellipsoid is shifted to zero-mean by converting the general ellipsoid polynomial equation to a translation-invariant form. In three dimensions the general form is

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + jz = 1 \; , \tag{10}$$

for which the parameters a to j are calculated by regression of at least nine signals. The parameters are converted to the following translation-invariant form

$$\begin{aligned} &\hat{a}(x - \hat{u})^2 + \hat{b}(y - \hat{v})^2 + \hat{c}(z - \hat{w})^2 \\ &+ \hat{d}(x - \hat{u})(y - \hat{v}) + \hat{e}(x - \hat{u})(z - \hat{w}) + \hat{f}(y - \hat{v})(z - \hat{w}) = 1. \end{aligned} \tag{11}$$

Calculation of $\hat{a}$ to $\hat{f}$ and $\hat{u}$, $\hat{v}$, $\hat{w}$ from the coefficients of Eq. (10) can be done in a computer algebra software by expansion of Eq. (11) and substitution of the constant term. The coefficients $\hat{a}$ to $\hat{f}$ are converted for the covariance matrix using Eqns. (9). The coefficient vector $(\hat{u}, \hat{v}, \hat{w})^T$ equals the center point of the ellipse and the synchronization offset of the receivers.

According to Theorem 1, the distances and angles in the tetrahedron of receivers are now directly characterized by the coefficients of the covariance matrix. The distances and angles are calculated by

$$\begin{aligned} d_2 &= \sqrt{3}\,\sigma_1 , & \cos(\varphi_3) &= \tfrac{\omega_1}{\sigma_1\sigma_2} , \\ d_3 &= \sqrt{3}\,\sigma_2 , & \cos(\varphi_4) &= \tfrac{\omega_2}{\sigma_1\sigma_3} , \\ d_4 &= \sqrt{3}\,\sigma_3 , & \cos(\theta) &= \tfrac{\omega_3}{\sigma_2\sigma_3} . \end{aligned}$$

3.3 Degenerate cases

When measurements δ_{ij} are corrupted by noise, or the far field assumption is violated, the solution of parameters in (10) might not yield an ellipsoid, but another type of quadric surface. For four receivers and nine transmitters, this constitutes a case of given measurements where there is no exact real solution to (2), as such time differences must lie on an ellipsoid by Theorem 1. Instead of using the regression scheme, one can obtain an approximation based on Theorem 1 by covariance estimation of the given time differences (x, y, z), denoted $\mathbf{\Sigma}^*$. Using $\hat{\mathbf{\Sigma}} = \frac{1}{3}\mathbf{\Sigma}^*$, distance and angle parameters can be estimated as in Section 3.2.

Other degenerate cases are when the ellipsoid is collapsed to a ellipse surface, or when transformed time differences in (11) lie on two intersecting quadric surfaces, thus giving infinite number of solutions.

4 Matrix factorization method

The matrix factorization method uses linear techniques to solve Problem 1 for receiver positions, transmitter directions and offsets. At least four receivers and nine transmitters are needed. Without loss of generality we assume that the solution is partially normalized for gauge freedom as the first receiver $\mathbf{r}_1 = \mathbf{0}$ and $f_1 = 0$, see Section 2.2.

Using the FFUTOA measurements δ_{ij}, collected in the matrix $\tilde{\mathbf{D}} = [\delta_{ij}]_{m\times k}$ we immediately obtain the unknowns g_j since $\delta_{1j} = \mathbf{r}_1^T\mathbf{n}_j + f_1 + g_j = g_j$, since $\mathbf{r}_1 = \mathbf{0}$ and $f_1 = 0$. We then subtract the first row containing g_j from all other rows of $\tilde{\mathbf{D}}$ and remove the first row of zeros to obtain a new matrix that fulfill

$$\mathbf{D}_2 = \begin{bmatrix}\mathbf{r}^T \ \mathbf{f}\end{bmatrix} \begin{bmatrix}\mathbf{n} \\ \vec{1}\end{bmatrix} \tag{12}$$

where $\vec{1}$ is a vector of ones. $\mathbf{D}_2$ is a product of two matrices of rank ≤ 4 and is thus itself of rank ≤ 4. This is used in [5]. Here we further reduce the rank of the factorization by subtracting the first column of $\mathbf{D}_2$ from all the other columns and remove the first row of columns. Both steps manipulating $\tilde{\mathbf{D}}$ can be done using the compaction matrices $\mathbf{C}_m$ of size $(m-1)\times m$ and $\mathbf{C}_k$ of size $k\times(k-1)$:

$$\mathbf{C}_m = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{bmatrix}, \ \mathbf{C}_k = \begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{13}$$

Then we have

$$\mathbf{D} = \mathbf{C}_m \tilde{\mathbf{D}} \mathbf{C}_k = \begin{bmatrix} \tilde{\mathbf{r}}^T & \mathbf{f} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{n}} \\ \vec{0} \end{bmatrix} = \tilde{\mathbf{r}}^T \tilde{\mathbf{n}}, \tag{14}$$

where $\tilde{\mathbf{r}}$ equals $\mathbf{r}$ with the first receiver removed as $\mathbf{r}_1 = \mathbf{0}$, and $\tilde{\mathbf{n}}$ is a $3\times(k-1)$ matrix with the j^{th} column $\tilde{\mathbf{n}}_j = \mathbf{n}_{j+1} - \mathbf{n}_1$. Now we have a rank-3 factorization, thus requiring at least four receivers and four transmitters. After applying SVD to $\mathbf{D} = \mathbf{U}\mathbf{S}\mathbf{V}^T$ we obtain the rank-3 factorization such that $\mathbf{D} = \bar{\mathbf{r}}^T\bar{\mathbf{n}}$ where $\bar{\mathbf{r}} = \mathbf{U}_3\mathbf{S}_3$ and $\bar{\mathbf{n}} = \mathbf{V}_3^T$. $\mathbf{U}_3,\ \mathbf{S}_3$ and $\mathbf{V}_3$ are the truncated parts of the SVD corresponding to the three largest singular values. This factorization of $\mathbf{D}$ is unique up to an unknown transformation $\mathbf{H}$ i.e. $\mathbf{D} = \bar{\mathbf{r}}^T\mathbf{H}^{-1}\mathbf{H}\bar{\mathbf{n}}$. We will find $\tilde{\mathbf{n}}_j = \mathbf{H}\bar{\mathbf{n}}$ i.e. $\mathbf{n}_{j+1} - \mathbf{n}_1 = \mathbf{H}\bar{\mathbf{n}}_j$ by using the constraints that

$$\begin{aligned} \tilde{\mathbf{n}}_j^T\tilde{\mathbf{n}}_j &= (\mathbf{n}_{j+1}-\mathbf{n}_1)^T(\mathbf{n}_{j+1}-\mathbf{n}_1) = 2 - 2\mathbf{n}_{j+1}^T\mathbf{n}_1 = 2 - 2(\mathbf{H}\bar{\mathbf{n}}_j + \mathbf{n}_1)^T\mathbf{n}_1 \\ &= -2\bar{\mathbf{n}}_j^T\mathbf{H}^T\mathbf{n}_1 = \bar{\mathbf{n}}_j^T\mathbf{H}^T\mathbf{H}\bar{\mathbf{n}}_j \ . \end{aligned} \tag{15}$$

We apply a change of variables with a 3×3 symmetric $\mathbf{C} = \mathbf{H}^T\mathbf{H}$ and a 3×1 vector $\mathbf{v} = \mathbf{H}^T\mathbf{n}_1$. From (15), we have the following equation for transmitter j:

$$\bar{\mathbf{n}}_j^T\mathbf{C}\bar{\mathbf{n}}_j + 2\bar{\mathbf{n}}_j^T\mathbf{v} = 0. \tag{16}$$

These equations are linear in the elements of $\mathbf{C}$ and $\mathbf{v}$ which have in total 9 variables. In general, with 8 such equations (thus 9 transmitters), we can solve this homogeneous linear equation system uniquely up to scale.

We can extract the solutions for $\mathbf{C}$ and $\mathbf{v}$ from the solution to the linear equation which is valid up to an unknown scaling factor and sign. We can determine the sign by using that $\mathbf{C}$ is positive definite and compute $\mathbf{H}$ by applying Cholesky factorization $\mathbf{C} = \mathbf{H}^T\mathbf{H}$. As $\mathbf{H}^T\mathbf{H} = \mathbf{H}^T\mathbf{R}^T\mathbf{R}\mathbf{H}$ for a rotation/mirroring matrix $\mathbf{R}$, this will give $\mathbf{H}$ uniquely up to $\mathbf{R}$. But as $\mathbf{R}$ corresponds to rotating/mirroring the coordinate system, $\mathbf{R}$ is a gauge freedom according to Section 2.2 and can be set to the identity matrix.

We can find the scale by using the constraint $\|\mathbf{n}_1\| = \|\mathbf{H}^{-T}\mathbf{v}\| = 1$. Note that fixing the scale in this way will also guarantee that $\mathbf{n}_j^T\mathbf{n}_j = (\mathbf{H}^T\bar{\mathbf{n}}_j + \mathbf{n}_1)^T(\mathbf{H}^T\bar{\mathbf{n}}_j + \mathbf{n}_1) = \underbrace{\bar{\mathbf{n}}_j^T\mathbf{H}^T\mathbf{H}\bar{\mathbf{n}}_j + 2\bar{\mathbf{n}}_j^T\mathbf{H}^T\mathbf{n}_1}_{=0 \text{ by } (15)} + \mathbf{n}_1^T\mathbf{n}_1 = \mathbf{n}_1^T\mathbf{n}_1 = 1$. Summarizing these steps yields Algorithm 1.

Algorithm 1. *Input*: FFUTOA measurement matrix $\tilde{\mathbf{D}}$ of size $(m = 4) \times (k = 9)$. *Output*: Receiver positions $\mathbf{r}$, transmitter directions $\mathbf{n}$, receiver offsets $\mathbf{f}$ and transmitter offsets $\mathbf{g}$. *Conditions*: (i) $\mathbf{D}$ must have rank 3, (ii) the linear equations (16) must only have a null space of dimension one, (iii) $\mathbf{C}$ must be positive definite.

1. Set $g_j := \tilde{\mathbf{D}}_{1j}$ and $\mathbf{D} := \mathbf{C}_m \tilde{\mathbf{D}} \mathbf{C}_k$ where $\mathbf{C}_l, \mathbf{C}_m$ is the compaction matrices in (13)
2. Calculate the SVD $\mathbf{D} = \mathbf{USV}^T$ and set $\bar{\mathbf{r}}$ to first three columns of $\mathbf{US}$ and $\bar{\mathbf{n}}$ to first three rows of $\mathbf{V}^T$
3. For the unknowns in the symmetric matrix $\mathbf{C}$ and vector $\mathbf{v}$, get the solution space for the equations $\bar{\mathbf{n}}_j^T \mathbf{C} \bar{\mathbf{n}}_j + 2\bar{\mathbf{n}}_j^T \mathbf{v} = 0$ where $\bar{\mathbf{n}}_j$ is the j^{th} column of $\bar{\mathbf{n}}$
4. Set the sign of the solution $\mathbf{C}, \mathbf{v}$ such that $C_{11} > 0$
5. Calculate the Cholesky decomposition $\mathbf{C} = \mathbf{H}^T \mathbf{H}$
6. Lock the scale of the solutions $\mathbf{H}, \mathbf{v}$ so that $\|\mathbf{H}^{-T}\mathbf{v}\| = 1$
7. Set $\mathbf{n}_1 := \mathbf{H}^{-T}\mathbf{v}$, $\mathbf{n}_{j+1} := \mathbf{H}\bar{\mathbf{n}}_j + \mathbf{n}_1$ and $\mathbf{r} := \mathbf{H}^{-T}\hat{\mathbf{r}}$

4.1 Degenerate cases

Theorem 2. *Degenerate cases for the minimal algorithm are when i) The transformed measurement matrix* $\mathbf{D}$ *has* $Rank(\mathbf{D}) \leq 2$ *or ii) The difference of the transmitter directions* $\mathbf{n}_j - \mathbf{n}_1$ *lie on the intersection of two or more quadric surfaces with constant term 0.*

Case i) happens iff receivers or transmitter directions lie in a plane. For ii), the case when the transmitter directions $\mathbf{n}_j$ *lie on the intersection of two or more a quadric surfaces is a special case.*

Proof: The only time the algorithm fails is when the prerequisites are not fulfilled. This happens iff i) $Rank(\mathbf{D}) \leq 2$ or ii) the linear equations (16) have a null space of dimension two or more.

For case i), step 2 will extract data from the SVD that are not uniquely determined from the measurements, but has several degrees of freedom. This will result in a reconstruction of $\mathbf{r}, \mathbf{n}$ that fulfills the measurements, but is not unique, as there are an infinite number of solutions.

$Rank(\mathbf{D}) \leq 2$ iff either receivers $\mathbf{r}$ or difference of transmitter directions $\mathbf{n}_j - \mathbf{n}_1$ in (14) are embedded in a lower dimensional subspace than assumed. Remembering that receiver positions can be translated as in Section 2.2, this is equivalent to receivers or transmitter directions being embedded in a plane.

For case (ii), there are at least two non linearly dependent solutions to (16). The solutions can be seen as constants for a quadric surface with radius 0 that $\bar{\mathbf{n}}_j$ should lie on, i.e.

$$\bar{\mathbf{n}}_j^T \mathbf{C}_1 \bar{\mathbf{n}}_j + \bar{\mathbf{n}}_j^T \mathbf{D}_1 = 0, \;\; \bar{\mathbf{n}}_j^T \mathbf{C}_2 \bar{\mathbf{n}}_j + \bar{\mathbf{n}}_j^T \mathbf{D}_2 = 0,$$

where $[\mathbf{C}_1 \; \mathbf{D}_1] \neq \lambda[\mathbf{C}_2 \; \mathbf{D}_2]$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $\mathbf{C}_i$ symmetric. As $\mathbf{n}_{j+1} - \mathbf{n}_1 = \mathbf{H}\bar{\mathbf{n}}_j$, this is equivalent to

$$\begin{aligned} (\mathbf{n}_j - \mathbf{n}_1)^T \mathbf{H}^{-T} \mathbf{C}_1 \mathbf{H}^{-1} (\mathbf{n}_j - \mathbf{n}_1) + (\mathbf{n}_j - \mathbf{n}_1)^T \mathbf{H}^{-T} \mathbf{D}_1 = 0, \\ (\mathbf{n}_j - \mathbf{n}_1)^T \mathbf{H}^{-T} \mathbf{C}_2 \mathbf{H}^{-1} (\mathbf{n}_j - \mathbf{n}_1) + (\mathbf{n}_j - \mathbf{n}_1)^T \mathbf{H}^{-T} \mathbf{D}_2 = 0, \end{aligned} \tag{17}$$

which is equivalent of the difference of the receiver directions $\mathbf{n}_j - \mathbf{n}_1$ lying on two or more quadric surfaces with constant term 0. As a special case, if the transmitter directions $\mathbf{n}_j$ lie on two or more different quadric surfaces, then the differences $\mathbf{n}_j - \mathbf{n}_1$ will fulfill (17). $\square$

Note that the degenerate cases characterized in i) is inherent to the problem, not the algorithm. There are fewer degrees of freedom to estimate than assumed, and thus there is not a unique solution. If both receivers and transmitter directions lie in the same plane, a similar algorithm for 2D based on the same factorization steps and equations can readily be constructed.

A special case is when $\mathbf{C}$ is not positive definite. Then there exists no real factorization $\mathbf{C} = \mathbf{H}^T\mathbf{H}$. There exists complex factorizations, e.g. obtained using eigenvalue decomposition $\mathbf{C} = \mathbf{Q}^T\mathbf{D}\mathbf{Q} = \mathbf{Q}^T\sqrt{\mathbf{D}}^T\sqrt{\mathbf{D}}\mathbf{Q} = \mathbf{H}^T\mathbf{H}$, which results in complex solutions. These cases equate exactly to the cases where the ellipsoid method does not get an ellipsoid from solving (10), as these are the cases where there are no exact real solutions to the given measurements.

5 Extension to overdetermined cases and noise

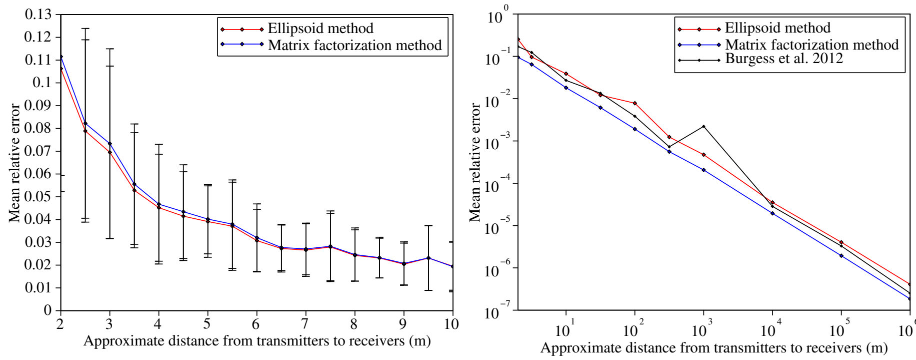
Both algorithms solve a minimal case, meaning that there are only a finite positive number of solutions to (2) given arbitrary measurements in general enough position. This can be seen from the fact that the matrix factorization algorithm does not lose any solutions from the solution space by any particular choice in any of the steps. Thus there one solution discounting gauge freedom. Another way of seeing it is by counting degrees of freedom. When using $m = 4$ receivers and $k = 9$ transmitters, the number of measurements $mk = 36$ equals to the number of unknowns, $4m + 3k - 7 = 36$ accounting for gauge freedom.

When having more than four receivers, more than nine transmitters and the measurements d_{ij} are not true FFUTOA measurements, due to noise or that the far field assumption does not hold, both methods can be extended in a straightforward manner.

For the ellipsoid method, two modifications are made. (i) When having more than nine receivers, the least squares solution to (11) can be calculated. (ii) When having more than four receivers, subproblems using only four receivers at a time are solved. With overlap of receivers used in the different subproblems, all distances between receivers can be calculated and multidimensional scaling [1] can be used to get the full coordinates of all receivers.

For the matrix factorization method, the three following modifications are made. (i) In step 2, the best rank 3 approximation can still be obtained by SVD, although $\mathbf{D}$ is not necessarily rank 3 . (ii) The system of equations in step 3 will in general only have the trivial solution, but is approximated to rank 8 by SVD to still attain the expected one dimensional solution set. (iii) $\|\mathbf{n}_j\|$ is only approximately 1, so $\mathbf{n}_j$ is normalized to be of length 1.

From here on, the extended methods will be used. Note that when only minimal number of measurements are available, the extended methods are equivalent to the minimal ones.



(a) Far field test, 4 receivers, 15 transmitters.

(b) Far field test, 5 receivers, 15 transmitters.

Fig. 2. Mean relative error of reconstructed receiver positions for 100 runs, plotted against the approximate distance from receivers to transmitters. (a) Bars are ± 1 standard deviation for the different transmitter distances.

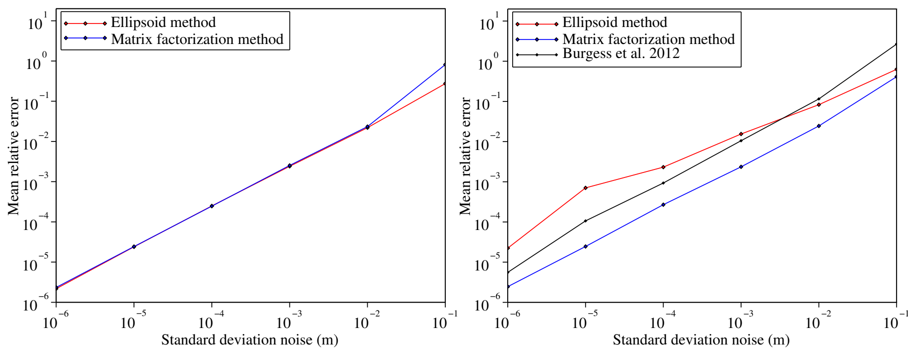
6 Experimental validation

To be able to evaluate the quality of a solution, receivers positions $\mathbf{r}_i$ are compared to ground truth receiver positions $\mathbf{r}_{i,gt}$. Receivers are rotated, mirrored and translated so that $\sum_i \|\mathbf{R}(\mathbf{r}_i - \mathbf{t}) - \mathbf{r}_{i,gt}\|^2$ is minimized, where $\mathbf{R}$ and $\mathbf{t}$ is a rotation/mirroring and translation respectively. Finding $\mathbf{R}$ and $\mathbf{t}$ is done by using [11]. For all experiments, relative errors are then defined as $\|\mathbf{r} - \mathbf{r}_{gt}\|_{Fro}/\|\mathbf{r}_{gt}\|_{Fro}$ where $\|\cdot\|_{Fro}$ is the Frobenius norm. All algorithms were implemented and run on a standard desktop computer in Scilab.

6.1 Simulations

For all simulations, offsets f_i and g_j are drawn from i.i.d. uniform distributions over $[0, 10]$. To evaluate the assumption that transmitters have a common direction to the receivers, transmitter positions $\mathbf{b}_j$ were uniformly distributed over a sphere of radius d. To be able to control how much further away transmitters were from receivers than the inter distance between receivers, four receivers were placed at a tetrahedron around the origin with side length 1m. As signal sources are often easily obtained in applications, 15 transmitters were used. UTOA Measurements were constructed as $\delta_{ij} = \|\mathbf{r}_i - \mathbf{b}_j\| + f_i + g_j$. The mean relative error for 100 runs each plotted against the transmitter distance d to the origin can be seen for different radii d in Fig. 2a for the minimal four receivers and in Fig. 2b for five receivers. The extra receiver was uniformly distributed in the cube of which the tetrahedron of the four first receivers were inscribed to.

Figure 2a shows that using only four receivers, both algorithms can get under 5% relative error with having transmitter approximately four times further away



(a) 4 receivers, 15 transmitters.

(b) 5 receivers, 15 transmitters.

Fig. 3. Measurements with additive Gaussian white noise. The standard deviation is plotted against the mean relative error of reconstructed receiver positions for 100 runs.

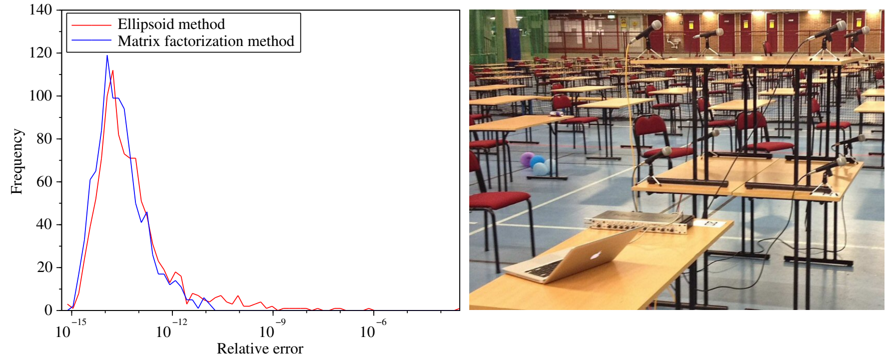
than the inter distance between receivers. For the experiment in Fig. 2b, we compare the results to the method in [5] as we now have the five receivers for the method to be applicable. The results indicate the ellipsoid method being slightly worse on short distances and the matrix factorization method being generally more accurate. Mean execution time was 8.0 ms, 2.1 ms and 30 ms for the ellipsoid method, matrix factorization method and the method in [5] each.

To test the robustness of the methods, white Gaussian noise was added to the measurements. The same setup as for the far field experiments was used, with transmitter distance of 10^7 from the receivers. In Fig. 3 relative error of reconstructed receiver positions are plotted against the standard deviation of the noise. The results indicate the ellipsoid method being slightly better with higher noise level when using the minimum four receivers, and the matrix factorization outperforming both the ellipsoid and the method in [5] using five receivers.

The numerical performance of the minimal algorithms were evaluated by generating problems where the measurement matrix $\tilde{\mathbf{D}}$ are FFUTOA (2). Receivers are drawn from i.i.d. uniform distributions in a cube of unit volume centered around the origin. Nine transmitter directions and four receivers were simulated. The error distribution for 1000 such experiments can be seen in Fig. 4a. Mean execution time for the ellipsoid method and the matrix factorization method was 3.2 and 1.9 ms respectively.

6.2 Real Data

The same data as in [5] was used, where the measurements d_{ij} were obtained from an experimental setup using eight SHURE SV100 microphones as receivers and random distinct manually made sounds as transmitters. The microphones were connected to a M-Audio Fast Track Ultra 8R audio interface. The 19 sound sources were approximately 30 m away from the receivers. Microphones were



(a) Minimal solver numerical performance.

(b) Experimental setup

Fig. 4. (a) Numerical performance of minimal solver in 1000 simulated experiments. (b) Setup for indoor experiment using microphones and distinct manually made sounds.

set in the corners of a cuboid of roughly $100 \times 105 \times 60\,\mathrm{cm}^3$. A picture of the experiment setup can be seen in Fig. 4b. The microphone offsets were created by adding uniformly i.i.d. silences between 0-1 s long to the beginning of each sound track, effectively starting the recordings at different unknown times. The beginning of each sound were matched by a heuristic cross correlation algorithm to create TDOA measurements.

As we have more than five microphones, the algorithms were also compared using the method in [5]. The mean reconstruction error on the microphone positions were 15 cm, 5 cm and 14 cm for the ellipsoid method, matrix factorization method and the method in [5] respectively. Most of the error are in the floor-to-roof direction. This can be explained by the sounds all being made close to ground level and thus the transmitter directions will be close to being in a plane, giving poor resolution in floor-to-roof direction.

7 Conclusions

We have presented two methods for solving the previously unsolved problem of sensor network calibration using only a minimal number of unsynchronized TDOA measurements in a far field setting. The assumption of far field signals is important, as the problem of trying to determine exact positions for transmitters is ill conditioned when the far field assumption is close to true.

Simulated experiments support the feasibility of the methods, and show that the minimal algorithms are numerically stable and fast, making them suitable in RANSAC schemes to weed out outliers. They also handle additive Gaussian noise well. The far field assumption gives good results as long as transmitter-receiver distances are four times larger than inter-receiver distances.

A comparison between the two methods, running on the minimal case of four receivers and nine transmitters, indicates the matrix factorization method being slightly faster and having better worst case precision than the ellipsoid method. The ellipsoid method however has a more plausible way of handling the case when the measurements are such that no exact real solutions exist, as per Section 3.3 and 4.1. The ellipsoid method estimates the covariance of the time differences for parameter estimation, whereas the matrix factorization finds a complex solution.

When having more than the minimum amount of four receivers and nine transmitters, the matrix factorization is easily extended to handle more than the minimal number of receivers and transmitters, and usually exhibit better average case performance than both the ellipsoid method and the method in [5], applicable when five or more receivers are available. The ellipsoid method is easily extended to handle more than the minimal nine transmitters, but not easily extended to handle more than the minimal four receivers. Although none of the methods are formally optimal in any sense, they are in closed form, fast, and can serve as initializations for further nonlinear optimization if need be. Both methods are significantly faster than the method in [5], and the matrix factorization method performs better in reconstructing the receiver positions.

In a real world experiment in an indoor environment, both methods perform well and the matrix factorization method reconstructs microphone positions with an average of 5 cm error from the previous 14 cm in [5].

Future work of interest is developing a method for calibration of UTOA networks not using the far field assumption, thus being able to solve problems when the far field assumption is far from true.

Acknowledgements The research leading to these results has received funding from the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) within the Research Training Group 1103 (Embedded Microsystems), the strategic research projects ELLIIT and eSSENCE, and Swedish Foundation for Strategic Research projects ENGROSS and VINST (grants no. RIT08-0075 and RIT08-0043).

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